MAXIMA

&

MINIMA

(KEY CONCEPTS + SOLVED EXAMPLES)

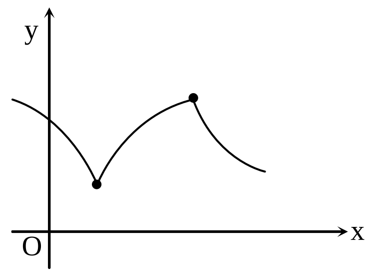
MAXIMA & MINIMA

1. *Maximum & Minimum Points*
2. *Conditions for maxima and minima of a function*
3. *Working rule for finding maxima and minima*
4. *Greatest & Least values of a function in a given interval*
5. *Properties of maxima & minima*
6. *Standard Geo. results related to maxima & minima*
7. *Important results*

KEY CONCEPTS

1. Introduction

In this chapter we shall study those points of the domain of a function where its graph changes its direction from upwards to downwards or from downwards to upwards. At such points the derivative of the function, if it exists, is necessarily zero.

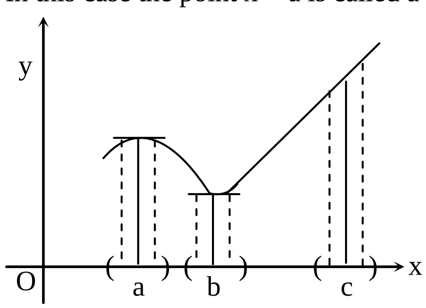


2. Maximum & Minimum Points

The value of a function f (x) is said to be maximum at x = a, if there exists a very small positive number h, such that

$f(x) < f(a)\ \forall\ x \in (a-h, a+h),\ x \neq a$

In this case the point x = a is called a point of maxima for the function f(x).



Similarly, the value of f(x) is said to the minimum at x = b, If there exists a very small positive number, h, such that

$f(x) > f(b), \forall\ x \in (b-h, b+h),\ x \neq b$

In this case x = b is called the point of minima for the function f(x).

Hene we find that,

(i) x = a is a maximum point of f(x)

$$\begin{cases} f(a) - f(a+h) > 0 \\ f(a) - f(a-h) > 0 \end{cases}$$

(ii) x = b is a minimum point of f(x)

$$\begin{cases} f(b) - f(b+h) < 0 \\ f(b) - f(b-h) < 0 \end{cases}$$

(iii) x = c is neither a maximum point nor a minimum point

$$\left\{ \begin{array}{l} f(c) - f(c+h) \\ \text{and} \\ f(c) - f(c-h) > 0 \end{array} \right\} \text{ have opposite signs.}$$

Note :

(i) The maximum and minimum points are also known as extreme points.

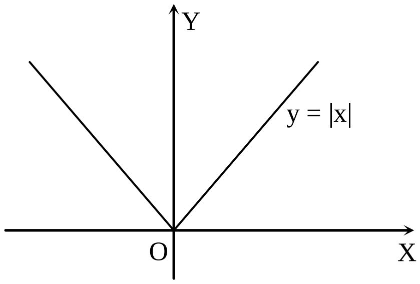
(ii) A function may have more than one maximum and minimum points.

(iii) A maximum value of a function f(x) in an interval [a,b] is not necessarily its greatest value in that interval. Similarly, a minimum value may not be the least value of the function. A minimum value may be greater than some maximum value for a function.

(iv) If a continuous function has only one maximum (minimum) point, then at this point function has its greatest (least) value.

(v) Monotonic functions do not have extreme points.

Ex. $f(x) = |x|$ has a minimum point at $x = 0$. It can be easily observed from its graph.



3. Conditions For Maxima & Minima of a Function

A. Necessary Condition : A point $x = a$ is an extreme point of a function f(x) if $f'(a) = 0$, provided $f'(a)$ exists. Thus if $f'(a)$ exists, then

$x = a$ is an extreme point $\Rightarrow f'(a) = 0$
or
$f'(a) \neq 0 \Rightarrow x = a$ is not an extreme point.

But its converse is not true i.e.

$f'(a) = 0$ $x = a$ is an extreme point.

For example if $f(x) = x^3$, then $f'(0) = 0$ but $x = 0$ is not an extreme point.

B. Sufficient Condition :

(i) The value of the function f(x) at $x = a$ is maximum, if $f'(a) = 0$ and $f''(a) < 0$.

(ii) The value of the function f(x) at $x = a$ in minimum if $f'(a) = 0$ and $f''(a) > 0$.

Note:

(i) If $f'(a) = 0$, $f''(a) = 0$, $f'''(a)$ 0 then $x = a$ is not an extreme point for the function f(x).

(ii) If $f'(a) = 0$, $f''(a) = 0$, $f'''(a) = 0$ then the sign of $f^{(iv)}(a)$ will determine the maximum and minimum value of function i.e. f(x) is maximum, if $f^{(iv)}(a) < 0$ and minimum if $f^{(iv)}(a) > 0$.

4. Working Rule For Finding Maxima & Minima

(I) Find the differential coefficient of f(x) with respect to x, i.e. $f'(x)$ and equate it to zero.

(ii) Find different real values of x by solving the equation $f'(x) = 0$. Let its roots be a, b, c,......

(iii) Find the value of $f''(x)$ and substitute the value of a, b, c ....in it and get the sign of $f'(x)$ for each value of x.

(iv) If $f'(a) < 0$ then the value of f (x) is maximum at $x = a$ and if $f'(a) > 0$ then value of f(x) will be minimum at $x = a$. Similarly by getting the signs of $f''(x)$ at other points b, c.... we can find the points of maxima and minima.

5. Greatest & Least Values of a Function in a Given Functional

If a function f(x) is defined in an interval [a, b], then greatest or least values of this function occurs either at x = a or x = b or at those values of x where $f'(x) = 0$.

Remember that a maximum value of the function f(x) in any interval [a, b] is not necessarily its greatest value in that interval. Thus greatest value of f(x) in interval [a, b]

$$= \max. [f(a), f(b), f(c)]$$

Least value of f(x) in interval [a, b]

$$= \text{Min.} [f(a), f(b), f(c)]$$

Where x = c is a point such that $f'(c) = 0$

6. Properties of Maxima & Minima

If f (x) is a continuous function and the graph of this function is drawn, then-

(i) Between two equal values of f(x), there lie at least one maxima or minima.

(ii) Maxima and minima occur alternately. For example if x = – 1,0,2,3 are extreme points of a continuous function and if x = 0 is a maximum point then x = – 1,2 will be minimum points.

(iii) When x passes a maximum point, the sign of $f'(x)$ changes from + ve to – ve, whereas x passes through a minimum point, the sign of $f'(x)$ changes from – ve to + ve.

(iv) If there is no change in the sign of dy/dx on two sides of a point, then such a point is not an extreme point.

(v) If f(x) is a maximum (minimum) at a point x = a, then 1/f (x), $[f(x) \neq 0]$ will be minimum (maximum) at that point.

(vi) If f(x) is maximum (minimum) at a point x = a, then for any $\lambda \in R$, $\lambda + f(x)$, log f(x) and for any k > 0, k f (x), $[f(x)]^k$ are also maximum (minimum) at that point.

7. Maxima & Minima of Functions of Two Variables

If a function is defined in terms of two variables and if these variables are associated with a given relation then by eliminating one variable, we convert function in terms of one variable and then find the maxima and minima by known methods.

8. Some Standard Geometrical Results Related to Maxima & Minima

The following results can easily be established.

(i) The area of rectangle with given perimeter is greatest when it is a square.

(ii) The perimeter of a rectangle with given area is least when it is a square.

(iii) The greatest rectangle inscribed in a given circle is a square.

(iv) The greatest triangle inscribed in a given circle is equilateral.

(v) The semi vertical angle of a cone with given slant height and maximum volume is $\tan^{-1}\sqrt{2}$.

(vi) The height of a cylinder of maximum volume inscribed in a sphere of radius a is a $2a/\sqrt{3}$.

9. Some Important Results

(i) Equilateral triangle :

Area = $(\sqrt{3}/4)\,x^2$, where x is its side.

(ii) Square :

Area = a^2 , perimeter = 4a,

where a is its side.

(iii) Rectangle:

Area = a b, perimeter = 2 (a + b)

where a, b are its sides

(iv) Trapezium :

Area = 1/2 (a+ b) h

Where a, b are lengths of parallel sides and h be the distance between them.

(v) Circle :

Area = πa^2 , perimeter = $2\pi a$,

where a is its radius.

(vi) Sphere :

Volume = $4/3\ \pi a^3$, surface $4\pi a^2$

where a is its radius

(vii) Right Circular cone :

Volume = 1/3 $\pi r^2 h$, curved surface = $\pi r \ell$

Where r is the radius of its base, h be its height and ℓ be its slant heights

(viii) Cylinder :

Volume = $\pi r^2 h$

whole surface = $2 \pi r (r + h)$

where r is the radius of the base and h be its height.

SOLVED EXAMPLES

Ex.1 $f(x) = 2x^3 - 21x^2 + 36x + 7$ has a maxima at -

(A) $x = 2$ (B) $x = 1$ (C) $x = 6$ (D) $x = 3$

Sol. $f'(x) = 6x^2 - 42x + 36$

$f''(x) = 12x - 42$

Now $f'(x) = 0 \Rightarrow 6(x^2 - 7x + 6) = 0$

$\Rightarrow x = 1, 6$

Also $f''(1) = 12 - 42 = -30 < 0$

$\therefore$ f(x) has a maxima at $x = 1$

Ans.[B]

Ex.2 The minimum value of the function x^x $(x > 0)$ is at -

(A) $x = 1$ (B) $x = e$

(C) $x = e^{-1}$ (D) None of these

Sol. Let $y = x^x \Rightarrow \log y = x \log x$

$\Rightarrow \frac{d}{dx}(\log y) = 1 + \log x$

and $\frac{d^2}{dx^2}(\log y) = \frac{1}{x} = x^{-1}$

Now for minimum value of y or log y

$\frac{d}{dx}(\log y) = 0 \Rightarrow 1 + \log x = 0$

$\Rightarrow x = e^{-1}$ Again for $x = e^{-1}$

$\frac{d^2}{dx^2}(\log y) = e > 0$

$\Rightarrow$ y is minimum at $x = e^{-1}$

Ans.[C]

Ex.3 If $x = p$ and $x = q$ are respectively the maximum and minimum points of the function $x^5 - 5x^4 + 5x^3 - 10$, then -

(A) $p = 0, q = 1$ (B) $p = 1, q = 0$

(C) $p = 1, q = 3$ (D) $p = 3, q = 1$

Sol. Let $f(x) = x^5 - 5x^4 + 5x^3 - 10$, then

$f'(x) = 5x^4 - 20x^3 + 15x^2$

$= 5x^2(x-1)(x-3)$

and $f''(x) = 20x^3 - 60x^2 + 30x$

For maxima and minima

$f'(x) = 0 \Rightarrow 5x^2(x-1)(x-3) = 0$

$\Rightarrow x = 0, 1, 3$ Also $f''(1) = -10 < 0$

$\Rightarrow x = 1$ is a point of maxima $\Rightarrow p = 1$

and $f''(3) = 90 > 0$

$\Rightarrow x = 3$ is a point of minima $\Rightarrow q = 3$.

Ans.[C]

Ex.4 Let x, y be two variables and $x > 0$, $xy = 1$. Then minimum value of $x + y$ is -

(A) 1 (B) 2 (C) 3 (D) 4

Sol. Let $A = x + y = x + 1/x$ ($\because xy = 1$)

$\Rightarrow \frac{dA}{dx} = 1 - \frac{1}{x^2}, \frac{d^2A}{dx^2} = \frac{2}{x^3}$

Now $\frac{dA}{dx} = 0 \Rightarrow x = 1, -1$

Also at $x = 1$, $\frac{d^2A}{dx^2} = 2 > 0$

$x = 1$ is a minimum point of A. So minimum value of $A = 1 + 1/1 = 2$.

Ans.[B]

Ex.5 The maximum value of function $\sin x(1 + \cos x)$ occurs at -

(A) $x = \pi/4$ (B) $x = \pi/2$

(C) $x = \pi/3$ (D) $x = \pi/6$

Sol. Let $f(x) = \sin x(1 + \cos x) = \sin x + \frac{1}{2}\sin 2x$,

then $f'(x) = \cos x + \cos 2x$

and $f''(x) = -\sin x - 2\sin 2x$

For maximum value $f'(x) = 0$

$\Rightarrow \cos x + \cos 2x = 0$

$\Rightarrow \cos x = -\cos 2x$

$\Rightarrow \cos x = \cos(\pi - 2x)$

$\Rightarrow x = \pi - 2x \Rightarrow x = \pi/3$

Again $f''(\pi/3) = -\sin(\pi/3) - 2\sin(2\pi/3)$

$= -\frac{3\sqrt{3}}{2} < 0$

$\Rightarrow$ Maximum value of function occurs at $x = \pi/3$

Ans.[C]

Ex.6 The maximum value of 3 sin x + 4 cos x is -

(A) 3 (B) 4 (D) 5

(D) 7

Sol. Let f (x) = 3 sin x + 4 cos x

$\Rightarrow$ f ' (x) = 3 cos x – 4 sin x

f '' (x) = – 3 sin x – 4 cos x

Now f ' (x) = 0 $\Rightarrow$ 3 cos x – 4 sin x = 0

$\Rightarrow$ tan x = 3/4

Also then sin x = 3/5, cos x = 4/5 and so at

x = $\tan^{-1}$ (3/4)

f '' (x) = –3(3/5) – 4(4/5) < 0

$\Rightarrow$ f(x) has a maxima at tan x = 3/4. Also its maximum value

= 3(3/5) + 4(4/5) = 5

Ans.[C]

Ex.7 If x = –1 and x = 2 are extreme points of the function y = a log x + bx^2 + x, then-

(A) a = 2, b = 1/2 (B) a = 2, b = – 1/2

(C) a = – 2, b = 1/2 (D) a = – 2, b = – 1/2

Sol. $\frac{dy}{dx} = \frac{a}{x}$ + 2 bx + 1

Since x = –1 and x = 2 are extreme points so dy/dx at these points must be zero. So

– a – 2b + 1 = 0 and a/2 + 4b + 1 = 0

$\Rightarrow$ a + 2b – 1 = 0 and a + 8b + 2 = 0

$\Rightarrow$ a = 2, b = – 1/2 **Ans.[B]**

Ex.8 In [0, 2π] one maximum value of x + sin 2x is -

(A) $\frac{2\pi}{3} + \frac{\sqrt{3}}{2}$ (B) $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

(C) $\frac{\pi}{3} + \frac{\sqrt{3}}{2}$ (D) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$

Sol. Let f (x) = x + sin 2x

$\Rightarrow$ f ' (x) = 1 + 2 cos 2x

f ''(x) = – 4 sin 2x

Now f ' (x) = 0 $\Rightarrow$ cos 2x = – 1/2

$\Rightarrow$ 2x = 2π/3, 4π/3,.....

$\Rightarrow$ x = π/3, 2π/3

But f '' (π/3) = – 4($\sqrt{3}$ /2) < 0

$\therefore$ f(x) is maximum at x = π/3 and its one maximum value

= π/3 + sin (2π/3)

= π/3 + $\sqrt{3}$ /2 **Ans.[C]**

Ex.9 The maximum and minimum values of sin 2x – x are-

(A) 1, –1 (B) $\frac{3\sqrt{3}-\pi}{6}, \frac{\pi-3\sqrt{3}}{6}$

(C) $\frac{\pi-3\sqrt{3}}{6}, \frac{3\sqrt{3}-\pi}{6}$ (D) Do not exist

Sol. f(x) = sin 2x – x

f ' (x) = 2 cos 2x – 1

f '' (x) = – 4 sin 2x

Now f ' (x) = 0 $\Rightarrow$ 2 cos 2x – 1 = 0

$\Rightarrow$ x = n π ± π/6 n = 0, 1,2, ...

$\Rightarrow$ x = π /6, 5 π/6, 7π /6,– π /6,.....

But f '' (π/6) = – 2 $\sqrt{3}$ < 0

$\Rightarrow$ x = π /6 is a max. point

Also f '' (5π /6) = 2 $\sqrt{3}$ > 0

$\Rightarrow$ x = 5π /6 is a min. point

Hence one max. value = f(π /6) = $\frac{3\sqrt{3}-\pi}{6}$

one min. value = f (5π/6) = – $\frac{3\sqrt{3}-5\pi}{6}$

But it is not there in given alternatives. Hence by alternate position another min. point is – π /6 so one min. value

= f (–π/6) = $\frac{\pi-3\sqrt{3}}{6}$ **Ans.[B]**

Ex.10 For what values of x, the function sinx + cos 2x (x>0) is minimum -

(A) $\frac{n\pi}{2}$ (B) $\frac{3(n+1)\pi}{2}$

(C) $\frac{(2n+1)\pi}{2}$ (D) None of these

Sol. Let f(x) = sin x + cos 2x, then

f ' (x) = cos x – 2 sin 2x

and f '' (x) = – sin x – 4 cos 2x

For minimum f'(x) = 0 $\Rightarrow$ cos x – 4 sin x cos x = 0

$\Rightarrow$ cos x (1– 4 sin x) = 0

$\Rightarrow$ cos x = 0 or 1– 4 sin x = 0 $\Rightarrow$ x = (2n + 1) π /2 or

$x = n\pi + (-1)^n \sin^{-1}\left(\frac{1}{4}\right), n \in Z$

Now $f''\left\{(2n+1)\frac{\pi}{2}\right\}$

$= -\sin\left\{(2n+1)\frac{\pi}{2}\right\} - 4\cos(2n+1)\pi$

$= -(-1)^n - 4(-1)^{2n+1} > 0$

The function is minimum at $x = \frac{(2n+1)\pi}{2}$

Ans.[C]

Ex.11 The minimum value of

$64 \sec x + 27 \text{ cosec } x, 0 < x < \pi/2$ is-

(A) 91 (B) 25

(C) 125 (D) None of these

Sol. Let $y = 64 \sec x + 27 \text{ cosec } x$

$\Rightarrow \frac{dy}{dx} = 64 \sec x \tan x - 27 \text{ cosec } x \cot x$

$\frac{d^2y}{dx^2} = 64 \sec^3 x + 64 \sec x \tan^2 x + 27 \text{ cosec}^3 x$

$+ 27 \text{ cosec } x \cot^2 x$

Now $\frac{dy}{dx} = 0 \Rightarrow 64 \sec x \tan x = 27 \text{ cosec } x \cot x$

$\Rightarrow \tan^3 x = 27/64$

$\Rightarrow \tan x = 3/4$

Also then $\frac{d^2y}{dx^2} > 0$ $(\because 0 < x < /2)$

So y is minimum when

$x = \tan^{-1}(3/4)$ and its

min. value $= 64(5/4) + 27(5/3) = 125$

Ans.[C]

Ex.12 If $0 \le c \le 5$, then the minimum distance of the point (0, c) from parabola $y = x^2$ is-

(A) $\sqrt{c-4}$ (B) $\sqrt{c-1/4}$

(C) $\sqrt{c+1/4}$ (D) None of these

Sol. Let $(\sqrt{t}, t)$ be a point on the parabola whose distance from (0, c), be d. Then

$z = d^2 = t + (t-c)^2 = t^2 + t(1-2c) + c^2$

$\Rightarrow \frac{dz}{dt} = 2t + 1 - 2c, \frac{d^2z}{dt^2} = 2 > 0$

Now $\frac{dz}{dt} = 0 \Rightarrow t = c - 1/2$

which gives the minimum distance. So

min. distance $= \sqrt{(c-1/2)+(-1/2)^2}$

$= \sqrt{c-1/4}$

Ans.[B]

Ex.13 The minimum value of the function

$\frac{40}{3x^4+8x^3-18x^2+60}$ is -

(A) 2/3 (B) 3/2

(C) 40/53 (D) None of these

Sol. Let $y = \frac{1}{40}(3x^4 + 8x^3 - 18x^2 + 60)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{40}(12x^3 + 24x^2 - 36x)$

and $\frac{d^2y}{dx^2} = \frac{1}{40}(36x^2 + 48x - 36)$

Now $\frac{dy}{dx} = 0 \Rightarrow x^3 + 2x^2 - 3x = 0$

or $x(x-1)(x+3) = 0$

or $x = 0, 1, -3$

At $x = 0$, $\frac{d^2y}{dx^2} = -36 < 0$

$\therefore$ y is maximum at x = 0

$\Rightarrow$ the given function i.e. 1/y is minimum at x = 0

$\therefore$ minimum value of the function

$\frac{40}{60} = \frac{2}{3}$

Ans.[A]

Ex.14 If $\frac{dy}{dx} = (x-1)^3 (x-2)^4$, then y is -

(A) maximum at x = 1

(B) maximum at x = 2

(C) minimum at x = 1

(D) minimum at x = 2

Sol. $\frac{dy}{dx} = 0 \Rightarrow x = 1, 2$. If $h > 0$ is very small number, then

at $x = 1–h$, $\frac{dy}{dx} = (–)\ (+) = –$ ve

$x = 1 + h$, $\frac{dy}{dx} = (+)\ (+) = +$ ve

at $x = 1$, $\frac{dy}{dx}$ changes its sign from –ve to + ve which shows that $x = 1$ is a minimum.

Ans.[C]

Ex.15 The maximum area of a rectangle of perimeter 176 cms. is -

(A) 1936 sq.cms. (B) 1854 sq.cms.

(C) 2110 sq.cms. (D) None of these

Sol. Let sides of the rectangle be x, y ; then

$2x + 2y = 176$ ...(1)

$\therefore$ Its area $A = xy = x\ (88– x)$

[form (1)] $= 88x – x^2$

$\Rightarrow \frac{dA}{dx} = 88 – 2x$, $\frac{d^2A}{dx^2} = – 2 < 0$

Now $\frac{dA}{dx} = 0 \Rightarrow x = 44$;

Also then $\frac{d^2A}{dx^2} < 0$. So area will be maximum when $x = 44$ and maximum area

$= 44 \times 44 = 1936$ sq. cms.

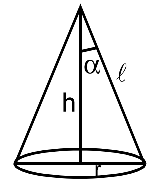
Ans.[A]

Ex.16 The semivertical angle of a right circular cone of given slant height and maximum volume is-

(A) $\tan^{–1} 2$ (B) $\tan^{–1} (\sqrt{2})$

(C) $\tan^{–1}\left(\frac{1}{2}\right)$ (D) $\tan^{–1}\left(\frac{1}{\sqrt{2}}\right)$

Sol. Let ℓ be the slant height and α be the semivertical angle of the right circular cone. Also suppose that h and r are its height and radius of the base.



Then $h = \ell \cos \alpha$, $r = \ell \sin \alpha$

Now volume $V = \frac{1}{3} \pi r^2 h$

$= \frac{1}{3} \pi \ell^3 \sin^2 \alpha \cos \alpha$

$\therefore \frac{dV}{d\alpha} = \frac{1}{3} = \pi \ell^3 [–\sin^3 \alpha + 2 \sin \alpha \cos^2 \alpha]$

$= \frac{1}{3} \pi \ell^3 [– \sin^3 \alpha + 2 \sin \alpha\ (1 – \sin^2 \alpha)]$

$= \frac{1}{3} \pi \ell^3 [2 \sin \alpha – 3 \sin^3 \alpha]$

$\therefore \frac{d^2V}{d\alpha^2} = \frac{1}{3} \pi \ell^3 [2 \cos \alpha – 9 \sin^2 \alpha \cos \alpha]$

Now $\frac{dV}{d\alpha} = 0 \Rightarrow \sin \alpha = 0$ or $2–3 \sin^2 \alpha = 0$

Now $\alpha \neq 0$ $\therefore 2 = 3 \sin^2\alpha$

or $2 \sin^2 \alpha + 2 \cos^2 \alpha = 3 \sin^2 \alpha$

or $\tan^2 \alpha = 2 \Rightarrow \tan \alpha = \sqrt{2}$

When $\tan \alpha = \sqrt{2}$, $\frac{d^2V}{d\alpha^2} < 0$

Thus when $\alpha = \tan^{–1} \sqrt{2}$, volume will be maximum. **Ans. [B]**

Ex.17 Two parts of 10 such that the sum of the twice of first with the square of second is minimum, are-

(A) 9, 1 (B) 5, 5 (C) 4, 6 (D) 1, 9

Sol. Let two parts be x and (10–x). If

$y = 2x + (10– x)^2$

Then $\frac{dy}{dx} = 2 – 2\ (10– x) = 2x – 18$

Now $\frac{dy}{dx} = 0\ x = 9$

Also then $\frac{d^2y}{dx^2} = 2 > 0$. Hence when x = 9, value of y is minimum. So required two parts of 10 are 9 and 1. **Ans.[A]**

Ex.18 For the curve $y = xe^x$ -

(A) x = 0 is a point of maxima

(B) x = 0 is a point of minima

(C) x = – 1 is a point of minima

(D) x = – 1 is a point of maxima

Sol. $y = xe^x \Rightarrow \frac{dy}{dx} = xe^x + e^x$

and $\frac{d^2y}{dx^2} = xe^x + 2e^x$

now $\frac{dy}{dx} = 0 \Rightarrow e^x (x + 1) = 0$

$\Rightarrow x = -1$ $[\because e^x > 0, \forall x]$

and at x = – 1, $\frac{d^2y}{dx^2} = e^{-x} (-1 + 2) > 0$

Therefore x = – 1 is a point of minima.

Ans.[C]

Ex.19 If sin x – x cos x is maximum at x = nπ, then-

(A) n is an odd positive integer

(B) n is an even negative integer

(C) n is an even positive integer

(D) n is an odd positive or even negative integer

Sol. Let f(x) = sin x – x cos x, then

$\Rightarrow f'(x) = \cos x - \cos x + x \sin x = x \sin x$

$f''(x) = x \cos x - \sin x$

Now $f'(x) = 0 \Rightarrow x \sin x = 0$

$\Rightarrow x = 0, n\pi$ n = 0, 1,2, ...

Also $f''(n\pi) = n\pi \cos n\pi - \sin n\pi$

$= (-1)^n n\pi$

But f(x) is maximum at x = nπ when $f''(n\pi) < 0$

$\Rightarrow (-1)^n n\pi < 0 \Rightarrow (-1)^n n < 0$

⇒ either n is an odd positive or even negative integer. **Ans.[D]**

Ex.20 $x (1 - x^2)$, $0 \le x \le 2$ is maximum at -

(A) x = 0 (B) x = 1

(C) $x = 1/\sqrt{3}$ (D) Nowhere

Sol. Let $y = x (1 - x^2)$

$\Rightarrow \frac{dy}{dx} = (1-x^2) - 2x^2 = 1 - 3x^2$

and $\frac{d^2y}{dx^2} = -6x$

Now $dy/dx = 0 \Rightarrow x = \pm \frac{1}{\sqrt{3}}$

Now at $x = \frac{1}{\sqrt{3}}$, $\frac{d^2y}{dx^2} < 0$.

Therefore y is maximum at $x = \frac{1}{\sqrt{3}}$

Ans.[C]

Ex.21 A curve whose slope at (x,y) is $x^2 - 2x$, passes through the point (2,0). The point with greatest ordinate on the curve is-

(A) (0, 0) (B) (0, 4)

(C) (0, 4/3) (D) (0,3/4)

Sol. Here $\frac{dy}{dx} = x^2 - 2x$

$\Rightarrow y = \frac{1}{3}x^3 - x^2 + c$

Since the curve passes through the point (2,0), therefore

$0 = (8/3) - 4 + c \Rightarrow c = 4/3$

∴ equation of curve $y = \frac{1}{3}x^3 - x^2 + \frac{4}{3}$ and

$\frac{dy}{dx} = x^2 - 2x$. $\frac{d^2y}{dx^2} = 2x - 2$

Now $\frac{dy}{dx} = 0 \Rightarrow x = 0, 2$

But at x = 0, $\frac{d^2y}{dx^2} = -2 < 0$

Thus at x = 0, y = 4/3 is maximum.

Ans.[C]

Ex.22 $f(x) = 1 + 2 \sin x + 3 \cos^2 x$ $(0 \le x \le 2\pi/3)$ is-

(A) minimum at x = π/2

(B) maximum at $x = \sin^{-1} (1/\sqrt{3})$

(C) minimum at x = π/3

(D) minimum at $x = \sin^{-1} (1/3)$

Sol. $f'(x) = 2 \cos x - 6 \cos x \sin x$

$f''(x) = -2 \sin x + 6 \sin^2 x - 6 \cos^2 x$

$= -2 \sin x + 12 \sin^2 x - 6$

Now $f'(x) = 0 \Rightarrow \cos x = 0$ and $\sin x = 1/3$

or x = π/2 & $x = \sin^{-1} (1/3)$

so $f''(\pi/2) = -2 + 12 - 6 > 0$

$$f''\left(\sin^{-1}\frac{1}{3}\right) = \frac{-2}{3}+\frac{4}{3}-6<0$$

$\therefore$ f(x) is minimum at x = $\pi/2$.

Ans.[A]

Ex.23 The minimum value of $e^{(2x^2-2x-1)\sin^2 x}$ is -

(A) e (B) 1/e (C) 1

(D) 0

Sol. Let y = $e^{(2x^2-2x-1)\sin^2 x}$

and u = $(2x^2 - 2x - 1)\sin^2 x$

Now $\frac{du}{dx}$

$= (2x^2 - 2x - 1)\, 2\sin x \cos x + (4x - 2)\sin^2 x$

$= \sin x\,[2(2x^2 - 2x)\cos x + (4x - 2)\sin x]$

$\frac{du}{dx} = 0 \Rightarrow \sin x = 0 \Rightarrow x = n\pi$

$\frac{d^2u}{dx^2} = \sin x \frac{d}{dx}\,[2(2x^2-2x-1)\cos x$

$+(4x-2)\sin x]+\cos x\,[2\cos x(2x^2-2x-1)$

$+ (4x-2)\sin x]$

At x = $n\pi$,

$\frac{d^2u}{dx^2} = 0 + 2\cos^2 n\pi(2n^2\pi^2 - 2n\pi - 1)>0$

Hence at x = $n\pi$, the value of u and so its corresponding the value of y is minimum and minimum value = e^0 = 1.

Ans.[C]