BINOMIAL THEOREM

(KEY CONCEPTS + SOLVED EXAMPLES)

BINOMIAL THEOREM

1. *Binomial Expressions*
2. *Binomial Theorem*
3. *Binomial Theorem for positive Integral Index*
4. *Number of Terms in the Expansion of (x + y + z)n*
5. *Middle Term in the Expansion of (x + a)n*
6. *Determination of Particular Term in the Expansion*
7. *Determination of Term from the End in Expansion of (x + a)n*
8. *Binomial Coefficients & Their Properties*
9. *Greatest Term in the Expansion of (x + a)n*
10. *Binomial Theorem for Any Index*

KEY CONCEPTS

1. Binomial Expressions

An algebraic expression containing two terms is called a **binomial expression**.

For example , 2x + 3, $x^2 - x/3$, x + a etc. are Binomial Expressions.

2. Binomial Theorem

The rule by which any power of a binomial can be expanded is called the **Binomial Theorem**.

3. Binomial Theorem for Positive Integral Index

If x and a are two real numbers and n is a positive integer then

$(x + a)^n = {}^nC_0\, x^n a^0 + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \ldots + {}^nC_r x^{n-r} a^r + \ldots + {}^nC_n x^0 a^n$.

Where ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ${}^nC_3$,......${}^nC_r$..... are called **binomial coefficients** which can be denoted by C_0, C_1, C_2, C_3, ...C_r......

3.1 General Term : In the expansion of $(x+a)^n$, $(r+1)^{th}$ term is called the **general term** which can be represented by T_{r+1}.

$$\mathbf{T_{r+1} = {}^nC_r\, x^{n-r}\, a^r}$$

$= {}^nC_r(\text{ first term})^{n-r}\ (\text{second term})^r$.

3.2 Characteristics of the expansion of $(x + a)^n$

Observing to the expansion of $(x + a)^n$, $n \in N$, we find that-

(i) The total number of terms in the expansion = (n + 1) i.e. one more than the index n.

(ii) In every successive term of the expansion the power of x (first term) decreases by 1and the power of (second term) increases by 1. Thus in every term of the expansion, the sum of the powers of x and a is equal to n (index).

(iii) The binomial coefficients of the terms which are at equidistant from the beginning and from the end are always equal i.e.

$${}^nC_r = {}^nC_{n-r}$$

Thus ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$,

${}^nC_2 = {}^nC_{n-2}$ etc.

(iv) ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$

3.3 Some deduction of Binomial Theorem :

(i) Expansion of $(x-a)^n$.

$(x - a)^n = {}^nC_0\, x^n a^0 - {}^nC_1 x^{n-1} a^1 + {}^nC_2 x^{n-2} a^2 -$

${}^nC_3 x^{n-3} a^3 + \ldots + (-1)^r\, {}^nC_r x^{n-r} a^r + \ldots + (-1)^n\, {}^nC_n\, x^o\, a^n$

This expansion can be obtained by putting (–a) in place of a in the expansion of $(x+a)^n$.

General term = $(r + 1)^{th}$ term

$$\mathbf{T_{r+1} = {}^nC_r(-1)^r .\ x^{n-r}\ a^r}$$

(ii) By putting x = 1 and a = x in the expansion of $(x + a)^n$, we get the following result $(1+x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2 x^2 + \ldots + {}^nC_r\, x^r + \ldots + {}^nC_n\, x^n$

which is the standard form of binomial expansion.

General term = (r + 1)th term

$$T_{r+1} = {}^nC_r\, x^r$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!}\,.\,x^r$$

(iii) By putting (–x) in place of x in the expansion of $(1+x)^n$

$(1-x)^n = {}^nC_0 - {}^nC_1\, x + {}^nC_2\, x^2 - {}^nC_3x^3 + \ldots\ldots + (-1)^r\, {}^nC_r x^r + \ldots\ldots + {}^nC_n x^n$.

General term = (r + 1)th term

$$T_{r+1} = (-1)^r.\, {}^nC_r\, x^r$$

$$= (-1)^r.\, \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!}\,.\,x^r$$

4. Number of Terms in the Expansion of $(x + y + z)^n$

$(x + y + z)^n$ can be expanded as-

$(x + y + z)^n = \{(x + y) + z\}^n$

$= (x + y)^n + {}^nC_1(x + y)^{n-1}.z + {}^nC_2(x + y)^{n-2} z^2 + \ldots\ldots + {}^nC_n z^n$.

$= (n + 1)$ terms + n terms + (n–1) terms + ...+ 1 term

$\therefore$ Total number of terms $= (n+1) + n + (n-1) + \ldots + 1$

$$= \frac{(n+1)(n+2)}{2}$$

5. Middle Term in the Expansion of $(x + a)^n$

(a) If n is even, then the number of terms in the expansion i.e. (n+1) is odd, therefore, there will be only one middle term which is $\left(\frac{n+2}{2}\right)^{th}$ term. i.e. $\left(\frac{n}{2}+1\right)^{th}$ term.

so middle term $= \left(\frac{n}{2}+1\right)^{th}$ term.

(b) If n is odd, then the number of terms in the expansion i.e. (n +1) is even, therefore there will be two middle terms which are

$= \left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ term.

Note : (i) When there are two middle terms in the expansion then their Binomial coefficients are equal.

(ii) Binomial coefficient of middle term is the greatest Binomial coefficient.

6. To Determine a Particular Term in the Expansion

In the expansion of $\left(x^\alpha \pm \frac{1}{x^\beta}\right)^n$, if x^m occurs in T_{r+1}, then r is given by

$$n\alpha - r(\alpha + \beta) = m$$

$$\Rightarrow\ r = \frac{n\alpha - m}{\alpha+\beta}$$

Thus in above expansion if constant term i.e. the term which is independent of x, occurs in T_{r+1} then r is determined by

$$n\alpha - r(\alpha + \beta) = 0$$

$$\Rightarrow r = \frac{n\alpha}{\alpha + \beta}$$

7. To Find a Term the end in the Expansion of $(x + A)^N$

It can be easily seen that in the expansion of $(x+a)^n$.

$(r+1)^{th}$ term from end = $(n-r+1)^{th}$ term from beginning.

i.e. $T_{r+1}(E) = T_{n-r+1}(B)$

$\therefore$ $T_r(E) = T_{n-r+2}(B)$

8. Binomial Coefficients & Their Properties

In the expansion of $(1+x)^n$; i.e. $(1+x)^n = {}^nC_0 + {}^nC_1 x + \dots\dots + {}^nC_r x^r + \dots + {}^nC_n x^n$

The coefficients ${}^nC_0$, ${}^nC_1$,${}^nC_n$ of various powers of x, are called binomial coefficients and they are written as

$C_0, C_1, C_2, \dots\dots, C_n$

Hence

$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_r x^r + \dots\dots + C_n x^n$ ...(1)

Where $C_0 = 1$, $C_1 = n$, $C_2 = \frac{n(n-1)}{2!}$

$C_r = \frac{n(n-1)\dots\dots(n-r+1)}{r!}$, $C_n = 1$

Now, we shall obtain some important expressions involving binomial coefficients-

(a) Sum of Coefficient : putting x = 1 in (1), we get

$C_0 + C_1 + C_2 + \dots\dots + C_n = 2^n$ ...(2)

(b) Sum of coefficients with alternate signs : putting x = –1 in(1)

We get

$C_0 - C_1 + C_2 - C_3 + \dots\dots = 0$ ...(3)

(c) Sum of coefficients of even and odd terms: from (3), we have

$C_0 + C_2 + C_4 + \dots\dots = C_1 + C_3 + C_5 + \dots\dots$ ...(4)

i.e. sum of coefficients of even and odd terms are equal.

from (2) and (4)

$\Rightarrow$ $C_0 + C_2 + \dots\dots = C_1 + C_3 + \dots\dots = 2^{n-1}$

(d) Sum of products of coefficients : Replacing x by 1/x in (1)

We get

$$\left(1 + \frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \dots + \frac{C_n}{x^n} + \dots$$

...(5)

Multiplying (1) by (5), we get

$$\frac{(1+x)^{2n}}{x^n} = (C_0 + C_1 x + C_2 x^2 + \dots)$$

$(C_0 + \frac{C_1}{x} + \frac{C_2}{x} + ...)$

Now, comparing coefficients of x^r on both the sides, we get

$C_0C_r + C_1C_{r+1} + ...+ C_{n-r} C_n = {}^{2n}C_{n-r}$

$= \frac{2n!}{(n+1)!(n-r)!}$ ...(6)

(e) Sum of squares of coefficients :

putting r = 0 in (6), we get

$C_0^2 + C_1^2 + C_2^2 + ...+ C_n^2 = \frac{2n!}{n!n!}$

(f) putting r = 1 in (6), we get

$C_0 C_1 + C_1C_2 + C_2C_3 + ...+ C_{n-1} C_n = {}^{2n}C_{n-1}$

$= \frac{2n!}{(n+1)!(n-1)!}$ ...(7)

(g) putting r = 2 in (6) , we get

$C_0C_2 + C_1C_3 + C_2C_4 +....+ C_{n-2}C_n = {}^{2n}C_{n-2}$

$= \frac{2n!}{(n+2)!(n-2)!}$ ... (8)

(h) Differentiating both sides of (1) w.r.t. x, we get

$n(1+ x)^{n-1} = C_1 + 2C_2x + 3C_3 x^2 +......+ nC_n x^{n-1}$

Now putting x = 1 and x = – 1 respectively

$C_1+ 2C_2 +3C_3+ .....+ nC_n = n.2^{n-1}$ .....(9)

and $C_1 - 2C_2 + 3C_3 - ....= 0$ ...(10)

(i) adding (2) and (9)

$C_0+ 2C_1 + 3C_2 + ...+ {}^{(n+1)}C_n = 2^{n-1}(n+2)$ ...(11)

(j) Integrating (1) w.r.t. x between the limits 0 to 1, we get,

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + ...+ \frac{C_nX^{n+1}}{n+1}\right]_0^1$$

$\Rightarrow C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ...+ \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$ ...(12)

Integrating (1) w.r.t. x between the limits –1 to 0, we get

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^0$$

$$= \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + ...+ \frac{C_nx^{n+1}}{n+1}\right]_{-1}^0$$

$\Rightarrow C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + ...+ \frac{(-1)^n.C_n}{n+1} = \frac{1}{(n+1)}$ ...(13)

9. Greatest Term in the Expansion of $(X + A)^N$

(a) The term in the expansion of $(x+a)^n$ of greatest coefficient

$$= \begin{cases} T_{\frac{n+2}{2}}, \text{ when } n \text{ is even} \\ T_{\frac{n+1}{2}}, T_{\frac{n+3}{2}}, \text{ when } n \text{ is odd} \end{cases}$$

(b) The greatest term

$$= \begin{cases} T_p \,\&\, T_{p+1} \text{ when } \frac{(n+1)a}{x+a} = p \in Z \\ T_{q+1} \text{ when } \frac{(n+1)a}{x+a} \notin Z \text{ and } q < \frac{(n+1)a}{x+a} < q+1 \end{cases}$$

10. Binomial Theorem For Any Index

When n is a negative integer or a fraction then the expansion of a binomial is possible only when

(i) Its first term is 1, and

(ii) Its second term is numerically less than 1.

Thus when $n \notin N$ and $|x| < 1$, then it states

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3$$

$$+ \ldots + \frac{n(n-1)(n-r+1)}{r!}x^r + \ldots \infty$$

10.1 General Term :

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!} \cdot x^r$$

Note :

(i) In this expansion the coefficient of different terms can not be expressed as ${}^nC_0$, ${}^nC_1$, ${}^nC_2$... because **n** is not a positive integer.

(ii) In this case there are infinite terms in the expansion.

10.2 Some Important Expansions :

If $|x| < 1$ and $n \in Q$ but $n \notin N$, then

(a) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots\ldots(n-r+1)}{r!}x^r + \ldots\ldots$

(b) $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots + \frac{n(n-1)\ldots\ldots(n-r+1)}{r!}(-x)^r + \ldots$

(c) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots + \frac{n(n+1)\ldots(n+r-1)}{r!}x^r + \ldots$

(d) $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots + \frac{n(n+1)\ldots(n+r-1)}{r!}(-x)^r + \ldots$

By putting n = 1, 2, 3 in the above results (c) and (d), we get the following results-

(e) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots + x^r + \ldots\ldots$

General term $T_{r+1} = x^r$

(f) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots(-x)^r + \ldots\ldots$

General term $T_{r+1} = (-x)^r$

(g) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots + (r+1)x^r + \ldots\ldots$

General term $T_{r+1} = (r+1)x^r$

(h) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots + (r+1)(-x)^r + \ldots\ldots$

General term $T_{r+1} = (r+1)(-x)^r$.

(i) $(1-x)^{-3} = 1+3x + 6x^2 + 10\,x^3 + \ldots\ldots + \frac{(r+1)(r+2)}{2!}x^r + \ldots\ldots$

General term = $\frac{(r+1)(r+2)}{2!}x^r$

(j) $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots\ldots + \frac{(r+1)(r+2)}{2!}(-x)^r + \ldots\ldots$

General term = $\frac{(r+1)(r+2)}{2!}(-x)^r$.

SOLVED EXAMPLES

Ex.1 The first four terms of the expansion of $\left(ax - \frac{1}{bx^2}\right)^5$ are-

(A) $a^5x^5 - 5\frac{a^4}{b}x^2 + 10\frac{a^3}{b^2x} - 10\frac{a^2}{b^3x^4}$

(B) $a^5x^5 + 5\frac{a^4}{b}x^2 - 10\frac{a^3}{b^2x} + 10\frac{a^2}{b^3x^4}$

(C) $a^5x^5 - 5\frac{a^4}{b}x^2 - 10\frac{a^3}{b^2x} - 10\frac{a^2}{b^3x^4}$

(D) $a^5x^5 + 5\frac{a^4}{b}x^2 + 10\frac{a^3}{b^2x} + 10\frac{a^2}{b^3x^4}$

Sol. $\left(ax - \frac{1}{bx^2}\right)^5$

$= {}^5C_0 (ax)^5 + {}^5C_1(ax)^4\left(-\frac{1}{bx^2}\right) +$

${}^5C_2 (ax)^3\left(-\frac{1}{bx^2}\right)^2 + {}^5C_3 (ax)^2\left(-\frac{1}{bx^2}\right)^3 +....$

$= a^5x^5 - 5\frac{a^4}{b}x^2 + 10\frac{a^3}{b^2x} - 10\frac{a^2}{b^3x^4} +........$

Ans. [A]

Ex.2 The sixth term in the expansion of $\left(3x^2 - \frac{1}{2x}\right)^8$ is-

(A) $\frac{189}{4}x$ (B) $-\frac{189}{4}x$

(C) $\frac{189}{4}x^2$ (D) $\frac{189}{4}x^3$

Sol. $T_6 = {}^8C_5 (3x^2)^3 \left(-\frac{1}{2x}\right)^5$

$= 56 \times (27x^6) \times \left(-\frac{1}{32x^5}\right)$

$= -\frac{189}{4}x$ **Ans. [B]**

Ex.3 If in the expansion of $(1+ y)^n$, the coefficient of 5^{th}, 6^{th} and 7^{th} terms are in A.P., then n is equal to-

(A) 7, 11 (B) 7, 14

(C) 8, 16 (D) None of these

Sol. As given ${}^nC_4$, ${}^nC_5$, ${}^nC_6$ are in AP.

$\Rightarrow {}^nC_4 + {}^nC_6 = 2.\,{}^nC_5$

$\Rightarrow \frac{n!}{(n-4)!\,4!} + \frac{n!}{(n-6)!\,6!} = 2\frac{n!}{(n-5)!\,5!}$

$\Rightarrow 30 + (n - 5)(n - 4) = 2.6\,(n - 4)$

$\Rightarrow n^2 - 21n + 98 = 0$

$\Rightarrow (n - 7)(n - 14) = 0$

$\therefore n = 7, 14$ **Ans. [B]**

Ex. 4 The sum of the coefficient of the terms of the expansion of polynomial $(1+ x- 3x^2)^{2143}$ is-

(A) 2^{2143} (B) 1

(C) –1 (D) 0

Sol. We get the sum of the coefficients of terms by putting x = 1 in the polynomial

$(1+ x- 3x^2)^{2143}$

$\therefore (1+1-3)^{2143} = (-1)^{2143}$

$= (-1)^{2142}.(-1)$

$= [(-1)^2]^{1021}.(-1)$

$= 1 \times -1 = -1.$

Ans. [C]

Ex.5 The middle term of the expansion $\left(x - \frac{2}{x}\right)^8$ is-

(A) 560 (B) –560

(C) 1120 (D) –1120

Sol. Since (n = 8) is even then there is only one middle term i.e. $T_{\frac{8+2}{2}} = T_5$

$\therefore T_5 = {}^8C_4(x)^4(-2/x)^4$

$= {}^8C_4.(-2)^4 = 16.{}^8C_4$

$= 1120$ **Ans. [C]**

Ex.6 The term independent from x in the expansion of $\left(\sqrt{x} - \frac{3}{x^2}\right)^{10}$ is -

(A) 3240 (B) – 3240

(C) 405 (D) – 405

Sol. Since we require term independent from x

$\therefore n\alpha - r(\alpha + \beta) = 0$

$\Rightarrow 10 \times \frac{1}{2} - r\left(\frac{1}{2} + 2\right) = 0$

$\Rightarrow r = 2$ i.e. 3^{rd} term.

$\therefore T_3 = {}^{10}C_2(\sqrt{x})^8(-3/x^2)^2$

$= {}^{10}C_2.(-3)^2.x^{o}$

$= \frac{10.9}{2.1}.9 = 405$ **Ans. [C]**

Ex.7 If in the expansion of $\left(x^3 - \frac{3}{x^2}\right)^{15}$ the r^{th} term is independent of x, then r equals-

(A) 8 (B) 9

(C) 10 (D) None of these

Sol. If r^{th} term is independent of x, then by the formula

$15 \times 3 - (r-1)(3+2) = 0$

$\Rightarrow$ $r - 1 = 9 \Rightarrow r = 10.$

Ans. [C]

Ex.8 If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$. then $C_0 + 2C_1 + 3C_2 + ... + (n+1)C_n$ is equal to-

(A) $2^{n-1}(n+2)$ (B) $2^n(n+1)$

(C) $2^{n-1}(n+1)$ (D) $2^n(n+2)$

Sol. Putting x = 1 in the given expansion, we get

$C_0 + C_1 + C_2 + C_3 + ...C_n = 2^n$ ...(1)

Now, differentiating the given expansion with respect to x and then putting x =1, we get

$C_1 + 2C_2 + 3C_3 + ... + nC_n = n.2^{n-1}$

...(2)

Given Exp.

$= C_0 + 2C_1 + 3C_2 + ... + (n+1)C_n$

$= (C_0 + C_1 + C_2 + ... + C_n)$

$+ (C_1 + 2C_2 + 3C_3 + ... + nC_n)$

$= 2^n + n.2^{n-1}$

[from (1) and (2)] $= 2^{n-1}(n+2)$

Ans. [A]

Ex.9 If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, then $\frac{(C_0+C_1)(C_1+C_2)...(C_{n-1}+C_n)}{C_1C_2...C_n}$ equals-

(A) $\frac{n^n}{(n+1)!}$ (B) $\frac{(n+1)^n}{n!}$

(C) $\frac{n^n}{n!}$ (D) None of these

Sol. The given expression

$= \frac{C_0C_1C_2....C_{n-1}}{C_1C_2....C_n}.\left(1+\frac{C_1}{C_0}\right)\left(1+\frac{C_2}{C_1}\right)$

$\left(1+\frac{C_3}{C_2}\right)...\left(1+\frac{C_n}{C_{n-1}}\right)$

$= \left(1+\frac{n}{1}\right)\left(1+\frac{n-1}{2}\right)\left(1+\frac{n-2}{3}\right)...\left(1+\frac{1}{n}\right)$

$(\because C_0 = C_n)$

$= \frac{(n+1)^n}{n!}$ **Ans. [B]**

Ex.10 In the expansion of $(4 - 3x)^7$, the numerically greatest term at x = 2/3 is -

(A) T_4 (B) T_5 (C) T_3 (D) T_2

Sol. $(4-3x)^7 = 4^7\left(1-\frac{3x}{4}\right)^7$

$\therefore \frac{T_{r+1}}{T_r} = \left|\frac{7-r+1}{r}.\frac{-3x}{4}\right|$

$= \frac{8-r}{2r}$ $\left(\because x = \frac{2}{3}\right)$

Now $T_{r+1} \geq T_{r+1}$ if $8 - r \geq 2r$

$\Rightarrow 3r \leq 8 \Rightarrow r \leq 2\frac{2}{3}$

$\therefore T_1 \leq T_2 \leq T_3 \geq T_4 \geq T_5.......$

$\therefore$ Numerical value of T_3 is greatest.

Ans. [C]

Ex.11 If |x| < 1/2, then expansion of $(1-2x)^{1/2}$ is-

(A) $1-x-\frac{1}{2}x^2$ .... (B) $1-x+\frac{1}{2}x^2$ ....

(C) $1+x-\frac{1}{2}x^2$ .... (D) None of these

Sol. $(1-2x)^{1/2} = 1+\frac{1}{2}(-2x)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-2x^2) + ...$

$= 1-x-\frac{1}{2}x^2.....$ **Ans. [A]**

Ex.12 The tenth term in the expansion of $(1+x)^{-3}$ is -

(A) $-55x^9$ (B) $55x^9$

(C) $-66x^{10}$ (D) $66x^{10}$

Sol. The tenth term of the expansion is

$$T_{10} = \frac{(-3)(-4)(-5)....(-3-8)}{9!}(x)^9$$

$$= \frac{-3(-4)(-5)....(-11)}{9!}x^9$$

$= -55x^9$ **Ans. [A]**

Ex.13 The value of $\sqrt{99}$ upto three decimals is -

(A) 9.949 (B) 9.958

(C) 9.948 (D) None of these

Sol. $\because \sqrt{99} = (100-1)^{1/2}$

$$= 10\left(1-\frac{1}{10^2}\right)^{1/2}$$

$$= 10\left[1-\frac{1}{2}.\frac{1}{10^2}+\frac{1/2.\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{10^2}\right)^2+....\right]$$

$= 10\,[1-0.005-0.0000125]$

$= 10\,[0.9949] = 9.949$

Ans. [A]

Ex.14 $1+\frac{1}{5}+\frac{1.3}{5.10}+\frac{1.3.5}{5.10.15}+....$ is equal to -

(A) $\frac{1}{\sqrt{5}}$ (B) $\frac{1}{\sqrt{2}}$

(C) $\sqrt{\frac{5}{3}}$ (D) $\sqrt{5}$

Sol. If $(1+x)^n = 1+\frac{1}{5}+\frac{1.3}{5.10}+\frac{1.3.5}{5.10.15}+....$

$$\left.\begin{array}{l}\text{then } nx = 1/5 \\ \frac{n(n-1)}{2!}x^2 = \frac{1.3}{5.10}\end{array}\right\} \Rightarrow n = -\frac{1}{2},\ x = -\frac{2}{5}$$

$\therefore S = [1+(-2/5)]^{-1/2}$

$= (3/5)^{-1/2} = \sqrt{\frac{5}{3}}$ **Ans. [C]**

Ex.15 The coefficient of x^4 in the expansion of $\frac{1+2x+3x^2}{(1-x)^2}$ is-

(A) 13 (B) 14

(C) 20 (D) 22

Sol. Exp. $= (1+2x+3x^2)(1-x)^{-2}$

$= (1+2x+3x^2)(1+2x+3x^2+4x^3+5x^4+...)$

$\therefore$ Coefficient of $x^4 = 5+8+9 = 22$

Ans. [D]

Ex.16 If the coefficients of rth and (r +1)th terms in the expansion of $(3+7x)^{29}$ are equal, then r equals-

(A) 15 (B) 21

(C) 14 (D) None of these

Sol. We have

$T_{r+1} = {}^{29}C_r 3^{29-r}(7x)^r = ({}^{29}C_r.3^{29-r}.7^r)\,x^r$

$\therefore a_r$ = coefficient of (r +1)th term

$= {}^{29}C_r.\,3^{29-r}.\,7^r$

Now, $a_r = a_{r-1}$

$\Rightarrow {}^{29}C_r.\,3^{29-r}.7^r = {}^{29}C_{r-1}.\,3^{30-r}.\,7^{r-1}$

$$\Rightarrow \frac{{}^{29}C_r}{{}^{29}C_{r-1}} = \frac{3}{7} \Rightarrow \frac{30-r}{7} = \frac{3}{7} \Rightarrow r = 21.$$

Ans. [B]

Ex.17 If the fourth term in the expansion of $(px+1/x)^n$ is 5/2 then the value of n and p are respectively-

(A) 6, 1/2 (B) 1/2, 6

(C) 3, 1 (D) 3, 1/2

Sol. The fourth term in expansion of $(px+1/x)^n$

$T_4 = {}^nC_3.(px)^{n-3}(1/x)^3 = 5/2.$

$\Rightarrow ({}^nC_3.p^{n-3}).\,x^{n-6} = 5/2.\,x^0$

Compairing the coefficient of x and constant term

$n - 6 = 0 \Rightarrow n = 6$

and $^nC_3 (p)^{n-3} = 5/2$

putting n = 6 in it

$6C_3\, p^3 = 5/2 \Rightarrow p^3 = 1/8 \Rightarrow p^3 = (1/2)^3$

$\Rightarrow p = 1/2$ **Ans. [A]**

Ex.18 The coefficient of x^4 in the expansion of $(1+ x + x^2 + x^3)^n$ is-

(A) $^nC_4$

(B) $^nC_4 + {^nC_2}$

(C) $^nC_1 + {^nC_2} + {^nC_4}.{^nC_2}$

(D) $^nC_4 + {^nC_2} + {^nC_1}.{^nC_2}$

Sol. Exp. $= (1+ x)^n (1+ x^2)^n$

$= (1+ {^nC_1}x + {^nC_2}x^2 + {^nC_3}x^3 + {^nC_4}x^4 + ...+ x^n)$

$(1+ {^nC_1}x^2 + {^nC_2}x^4 + ...+ x^{2n})$

$\therefore$ Coefficient of $x^4 = {^nC_4} + {^nC_2}.{^nC_1} + {^nC_2}$

Ans. [D]

Ex. 19 If $(2- x - x^2)^{2n} = a_0 + a_1x + a_2x^2 + a_3x^3 + ...$, then the value of $a_0 + a_2 + a_4 + ...$ is-

(A) 2^{n-1} (B) 2^{2n}

(C) 2^{2n-1} (D) None of these

Sol. Putting x = 1 and x = –1 in the given expansion, we get

$a_0 + a_1 + a_2 + a_3 + a_4 + ... = 0$

$a_0 - a_1 + a_2 - a_3 + a_4 - ....= 2^{2n}$

Adding $2(a_0 + a_2 + a_4 + ...) = 2^{2n}$

$\Rightarrow a_0 + a_2 + a_4 + ... = 2^{2n-1}$

Ans. [C]

Ex.20 $(x+\sqrt{x^3-1})^5 + (x-\sqrt{x^3-1})^5$ is a polynomial of the order of -

(A) 5 (B) 6 (C) 7 (D) 8

Sol. $(x+\sqrt{x^3-1})^5 + (x-\sqrt{x^3-1})^5$

$= 2\,[x^5 + 5C_2.x^3 (x^3-1) + 5C_4\, x(x^3-1)^2]$

$= 2\,[x^5 + 10x^3 (x^3-1) + 5x(x^6 - 2x^3 + 1)]$

$= 10x^7 + 20x^6 + 2x^5 - 20x^4 - 20x^3 + 10x$

$\therefore$ polynomial has order of 7.

Ans. [C]

Ex.21 If x^m occurs in the expansion of $\left(x+\frac{1}{x^2}\right)^{2n}$, the coefficient of x^m is -

(A) $\frac{(2n)!}{m!(2n-m)!}$

(B) $\frac{(2n)!\;3!\,3!}{(2n-m)!}$

(C) $\frac{(2n)!}{\left(\frac{2n-m}{3}\right)!\left(\frac{4n+m}{3}\right)!}$

(D) None of these

Sol. The general term in the expansion of the given expression is

$T_{r+1} = {^{2n}C_r}\, x^{2n-r} \left(\frac{1}{x^2}\right)^r = {^{2n}C_r}\, x^{2n-3r}$

For the coefficient of x^m, we must have

$2n - 3r = m \Rightarrow r = \frac{2n-m}{3}$

So, coefficient of x^m

$= {^{2n}C_{\frac{2n-m}{3}}} = \frac{(2n)!}{\left(\frac{2n-m}{3}\right)!\left(\frac{4n+m}{3}\right)!}$

Ans. [C]

Ex.22 If the third term in the expansion of $\left[x + x^{\log_{10} x}\right]^5$ is equal to 10,00,000, then x equals-

(A)10 (B) 10^2

(C)10^3 (D) No such x exists

Sol. Here $T_3 = {^5C_2}x^3\, (x^{\log_{10} x})^2 = 10^6$

or $x^3\, x^{2\log_{10} x} = 10^5$

Taking log of both sides, we get

$3 \log_{10} x + 2 (\log_{10} x)^2 = 5$

or $2(\log_{10}x)^2 + 5 \log_{10} x - 2 \log_{10} x - 5 = 0$

or $(\log_{10} x - 1)(2 \log_{10} x + 5) = 0$

or $x = 10$ or $2 \log_{10} x + 5 = 0$

Ans. [A]

Ex.23 The greatest integer in the expansion of $(1+ x)^{2n+2}$ is-

(A) $\frac{(2n)!}{(n!)^2}$ (B) $\frac{(2n+2)!}{[(n+1)!]^2}$

(C) $\frac{(2n+2)!}{n!(n+1)!}$ (D) $\frac{(2n)!}{n!(n+1)!}$

Sol. The coefficient of $(r+1)^{th}$ term in the expansion of $(1+x)^{n+2}$ will be maximum.

If $r \le \frac{(2n+2)+1}{2}$

$r \le (n+1) + 1/2$

$r = n + 1$

= Maximum coefficient = ${}^{2n+2}C_{n+1}$

$= \frac{(2n+2)!}{(n+1)!(n+1)!}$

$= \frac{(2n+2)!}{[(n+1)!]^2}$ **Ans. [B]**

Ex.24 The greatest integer which divides $101^{100} - 1$ is

(A) 100 (B) 1000

(C) 10,000 (D) 100,000

Sol. $101^{100} - 1 = (100+1)^{100} - 1$

$= 100^{100} + {}^{100}C_1\, 100^{99} + {}^{100}C_2\, 100^{98} + ... + 1 - 1$

$= 100^{100} + {}^{100}C_1 100^{99} + {}^{100}C_2\, 100^{98} + ... + {}^{100}C_{99}\, 100^1$

$= 100(100^{99} + {}^{100}C_1\, 100^{98} + ... + {}^{100}C_{99})$

$= 100\,(100^{99} + {}^{100}C_1 100^{98} + ... + {}^{100}C_{98}\, 100 + {}^{100}C_{99})$

$= 100(100^{99} + {}^{100}C_1 100^{98} + ... + {}^{100}C_{98} 100 + 100)$

$= 100^2\,(100^{98} + {}^{100}C_1\, 100^{97} + ... + {}^{100}C_2 + 1)$

$\therefore$ the greatest integer which divides given number $= 100^2 = 10,000$

Ans.[C]

Ex.25 The sum of the rational terms in the expansion of $(\sqrt{2} + 3^{1/5})^{10}$ is equal to

(A) 40 (B) 41 (C) 42 (D) 0

Sol. Here $T_{r+1} = {}^{10}C_r\,(\sqrt{2})^{10-r}\,(3^{1/5})^r$,

where r = 0, 1, 2, ...,10.

We observe that in general term T_{r+1} powers of 2 and 3 are $\frac{1}{2}(10-r)$ and $\frac{1}{5}r$ respectively and $0 \le r \le 10$. So both these powers will be integers together only when r = 0 or 10

$\therefore$ sum of required terms

$= T_1 + T_{11}$

$= {}^{10}C_0(\sqrt{2})^{10} + {}^{10}C_{10}\,(3^{1/5})^{10}$

= 32 + 9 = 41

Ans. [B]

Ex .26 The coefficient of the term independent of x in the expansion of $(1+x+2x^3)\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ is-

(A) 1/3 (B) 19/54 (C) 17/54 (D) 1/4

Sol. $(1+x+2x^3)\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$

$= (1+x+2x^3)\left[\sum_{r=0}^{9} {}^9C_r\left(\frac{3}{2}x^2\right)^{9-r}\left(-\frac{1}{3x}\right)^r\right]$

$= (1 + x + 2x^3) =$

$+ \left[\sum_{r=0}^{9} {}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r x^{19-3r}\right] +$

$2\left[\sum_{r=0}^{9} {}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r x^{21-3r}\right]$

Clearly, first and third expansions contain term independent of x and are obtained by equation 18 – 3r = 0 and 21–3 r = 0 respectively.

So, coefficient of the term independent of

$x = {}^9C_6\left(\frac{3}{2}\right)^{9-6}\left(-\frac{1}{3}\right)^6 + 2$

$\left({}^9C_7\left(\frac{3}{2}\right)^{9-7} - \left(\frac{1}{3}\right)^7\right) = \frac{7}{18} - \frac{7}{27} = \frac{17}{54}$

Ans. [C]

Ex.27 If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, then $3C_0 - 5C_1 + 7C_2 + ... + (-1)^n(2n+3)\,C_n$ equals-

(A) 1 (B) $2(2n+3)\,2^n$

(C) $(2n+3)\,2^{n-1}$ (D) 0

Sol. We have

$3C_0 - 5C_1 + 7\,C_2 + ... + (-1)^n\,(2n+3)\,C_n$

$= 3C_0 - 3C_1 + 3C_2 + ... + (-1)^n\,3C_n - 2C_1 + 4C_2 + ... + (-1)^n\;2n\,C_n$

$= 3(C_0 - C_1 + C_2 + ... + (-1)^n\,C_n)$
$\quad -2(C_1 - 2C_2 + ... (-1)^n\,nC_n)$

$= 3 \times 0 - 2 \times 0 = 0.$ **Ans. [D]**

Ex.28 If the sum of the coefficients in the expansion of $(1+2x)^n$ is 6561, the greatest term in the expansion for x = 1/2 is -

(A) 4^{th} (B) 5^{th}

(C) 6^{th} (D) None of these

Sol. Sum of the coefficients in the expansion of

$\Rightarrow (1+2x)^n = 6561$

$\Rightarrow (1+2x)^n = 6561$ when x = 1

$\Rightarrow 3^n = 6561 \Rightarrow 3^n = 3^8 \Rightarrow n = 8$

Now, $\frac{T_{r+1}}{T_r} = \frac{^8C_r(2x)^r}{^8C_{r-1}(2x)^{r-1}} = \frac{9-r}{r} . 2x$

$\Rightarrow \frac{T_{r+1}}{T_r} = \frac{9-r}{r}$ $[\because x = 1/2]$

$\therefore \frac{T_{r+1}}{T_r} > 1 \Rightarrow \frac{9-r}{r} > 1$

$\Rightarrow 9 - r > r \Rightarrow 2r < 9 \Rightarrow r < 4\frac{1}{2}$

Hence, 5^{th} term is the greatest term.

Ans. [B]

Ex.29 If $(r+1)^{th}$ term is $\frac{3.5...(2r-1)}{r!}\left(\frac{1}{5}\right)^r$, then this is the term of binomial expansion-

(A) $\left(1-\frac{2}{5}\right)^{1/2}$ (B) $\left(1-\frac{2}{5}\right)^{-1/2}$

(C) $\left(1+\frac{2}{5}\right)^{-1/2}$ (D) $\left(1+\frac{2}{5}\right)^{1/2}$

Sol. $T_{r+1} = \frac{3.5...(2r-1)}{r!}\left(\frac{1}{5}\right)^r$

$= \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)....\left(\frac{2r-1}{2}\right)}{r!}\left(\frac{2}{5}\right)^r$