CIRCLE

(KEY CONCEPTS + SOLVED EXAMPLES)

══CIRCLE══

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KEY CONCEPTS

1. Definition

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point is always constant. The fixed point is called the centre and constant distance is called the radius of the circle.

NOTE :

(i) If r (r > 0) is the radius of a circle, the diameter d = 2r is the maximum distance between any two points on the given circle

(ii) The length of the curve or perimeter (also called circumference) of circle is = $2\pi r$ or πd

(iii) The area of circle = πr^2 or

(iv) Lines joining any two points of a circle is called chord of circle

(v) Curved section of any two point of a circle is called arc of circle.

(vi) Angle subtended at the centre of a circle by any arc is given by = arc/radius.

(vii) Angle subtended at the centre of a circle by an arc is double of angle subtended at the circumference of a circle.

2. Standard forms of Equation of a Circle

2.1 General Equation of a Circle :

The general equation of a circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$, Where g, f, c are constants.

(i) Centre of a general equation of a circle is (–g, –f)

i.e. $\left(-\frac{1}{2}\text{ coefficient of x}, -\frac{1}{2}\text{ coefficient of y}\right)$

(ii) Radius of a general equation of a circle is

$$\sqrt{g^2 + f^2 - c}$$

NOTE :

(i) The general equation of second degree $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a circle if $a = b \neq 0$ and $h = 0$.

(ii) Locus of a point P represent a circle if its distance from two points A and B is not equal i.e. PA = kPB represent a circle if $k \neq 1$

(iii) General equation of a circle represents –

(a) A real circle if $g^2 + f^2 - c > 0$

(b) A point circle if $g^2 + f^2 - c = 0$

(c) An imaginary circle if $g^2 + f^2 - c < 0$

(iv) In General equation of a circle –

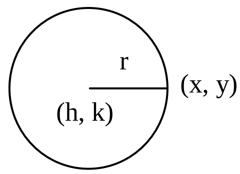
(a) If $c = 0 \Rightarrow$ The circle passes through origin

(b) If $f = 0 \Rightarrow$ The centre is on x – axis

(c) If $g = 0 \Rightarrow$ The centre is on y – axis

2.2 Central Form of Equation of a circle :

The equation of a circle having centre (h, k)and radius r is $(x - h)^2 + (y - k)^2 = r^2$



NOTE :

(i) If the centre is origin, then the equation of the circle is $x^2 + y^2 = r^2$

(ii) If r = 0 than circle is called point circle and its equation is

$(x - h)^2 + (y - k)^2 = 0$

2.3 Diametral Form :

If (x_1, y_1) and (x_2, y_2) be the extremities of a diameter, then the equation of circle is

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

2.4 Parametric Equation of a Circle :

(i) The parametric equations of a circle $x^2 + y^2 = r^2$ are $x = r\cos\theta$, $y = r\sin\theta$. Hence parametric coordinates of any point lying on the circle $x^2 + y^2 = r^2$ are $(r\cos\theta, r\sin\theta)$.

(ii) The parametric equations of the circle $(x - h)^2 + (y - k)^2 = r^2$ are $x = h + r\cos\theta$, $y = k + r\sin\theta$. Hence parametric coordinates of any point lying on the circle are $(h + r\cos\theta, k + r\sin\theta)$

(iii) Parametric equations of the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is

$x = -g + \cos\theta$

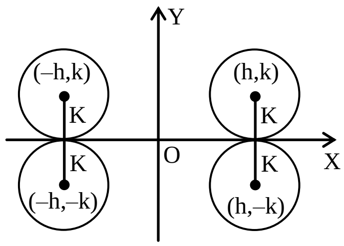
$y = -f + \sin\theta$

3. Equation of a Circle in some special cases

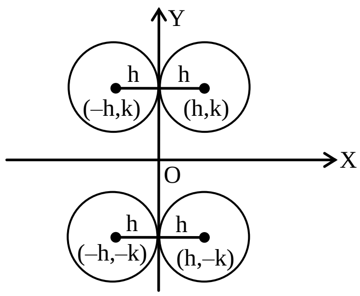
(i) If centre of circle is (h, k) and passes through origin then its equation is $(x - h)^2 + (y - k)^2 = h^2 + k^2$

$\Rightarrow x^2 + y^2 - 2hx - 2ky = 0$

(ii) If the circle touches x axis then its equation is (Four cases) $(x \pm h)^2 + (y \pm k)^2 = k^2$



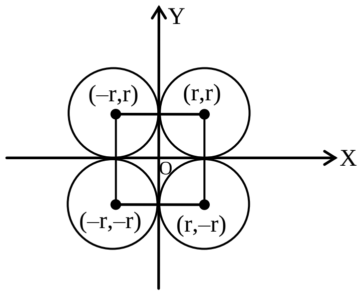
(iii) If the circle touches y axis then its equation is (Four cases)



$(x \pm h)^2+(y \pm k)^2 = h^2$

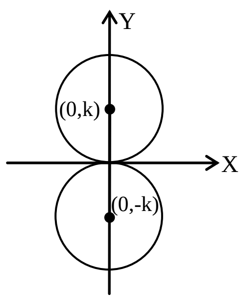
(iv) If the circle touches both the axis then its equation is (Four cases)

$(x \pm r)^2 + (y \pm r)^2 = r^2$



(v) If the circle touches x axis at origin (Two cases)

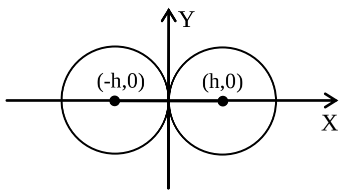
$x^2 + (y \pm k)^2 = k^2 \Rightarrow x^2 + y^2 \pm 2ky = 0$



(vi) If the circle touches y axis at origin (Two cases)

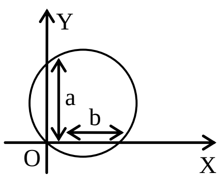
$(x \pm h)^2 + y^2 = h^2$

$\Rightarrow x^2 + y^2 \pm 2xh = 0$



(vii) If the circle passes through origin and cut intercept of a and b on axes, the equation of circle is (Four cases)

$x^2 + y^2 - ax - by = 0$ and centre is $(a/2, b/2)$



4. Position of a Point with respect to a Circle

A point (x_1, y_1) lies outside, on or inside a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ according as

$S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ is positive, zero or negative i.e.

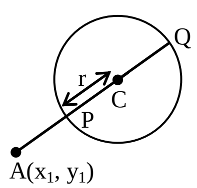
$S_1 > 0 \Rightarrow$ Point is outside the circle.

$S_1 = 0 \Rightarrow$ Point is on the circle.

$S_1 < 0 \Rightarrow$ Point is inside the circle.

4.1 The least and greatest distance of a point from a circle :

Let $S = 0$ be a circle and $A(x_1, y_1)$ be a point. If the diameter of the circle which is passing through the circle at P and Q. then



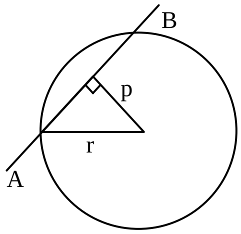
$AP = AC - r$ = least distance

$AQ = AC + r$ = greatest distance,

where 'r' is the radius and C is the centre of circle

5. Line and Circle

Let $L = 0$ be a line and $S = 0$ be a circle, if 'r' be the radius of a circle and p be the length of perpendicular from the centre of circle on the line, then if



$p > r \Rightarrow$ Line is outside the circle

$p = r \Rightarrow$ Line touches the circle

$p < r \Rightarrow$ Line is the chord of circle

$p = 0 \Rightarrow$ Line is diameter of circle

NOTE :

(i) Length of the intercept made by the circle on the line is $= 2\sqrt{r^2 - p^2}$

(ii) The length of the intercept made by line $y = mx + c$ with the circle $x^2 + y^2 = a^2$ is

$$2\sqrt{\frac{a^2(1+m^2)-c^2}{1+m^2}}$$

5.1 Condition of Tangency :

A line L = 0 touches the circle S = 0, if length of perpendicular drawn from the centre of the circle to the line is equal to radius of the circle i.e. p = r. This is the condition of tangency for the line L = 0

Circle $x^2 + y^2 = a^2$ will touch the line $y = mx + c$ if $c = \pm a\sqrt{1+m^2}$

Again

(a) If $a^2(1 + m^2) - c^2 > 0$ line will meet the circle at real and different points.

(b) If $c^2 = a^2(1 + m^2)$ line will touch the circle.

(c) If $a^2(1 + m^2) - c^2 < 0$ line will meet circle at two imaginary points.

5.2 Intercepts made on coordinate axes by the circle :

The intercept made by the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ on -

(i) x axis $= 2\sqrt{g^2 - c}$

(ii) y axis $= 2\sqrt{f^2 - c}$

NOTE : Circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts

(i) x axis in two real, coincident or imaginary points according as $g^2 >, =, < c$

(ii) y axis in two real, coincident or imaginary points according as $f^2 >, =, < c$

6. Equation of Tangent & Normal

6.1 Equation of Tangent :

The equation of tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at a point (x_1, y_1) is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ or $T = 0$

NOTE :

(i) The equation of tangent to circle $x^2 + y^2 = a^2$ at point (x_1, y_1) is $xx_1 + yy_1 = a^2$

(ii) **Slope Form:** From condition of tangency for every value of m, the line $y = mx \pm a\sqrt{1+m^2}$ is a tangent of the circle $x^2 + y^2 = a^2$ and its point of contact is $\left(\frac{\mp am}{\sqrt{1+m^2}}, \frac{\pm a}{\sqrt{1+m^2}}\right)$

6.2 Equation of Normal :

Normal to a curve at any point P of a curve is the straight line passes through P and is perpendicular to the tangent at P.

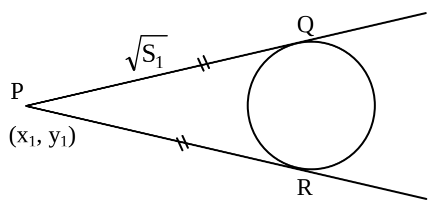
The equation of normal to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ at any point (x_1, y_1) is

$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$$

6.3 Length of Tangent :

From any point, say $P(x_1, y_1)$ two tangents can be drawn to a circle which are real, coincident or imaginary according as P lies outside, on or inside the circle.



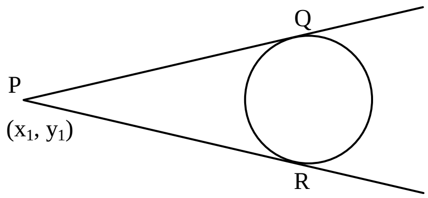
Let PQ and PR be two tangents drawn from $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Then PQ = PR is called the length of tangent drawn from point P and is given by $PQ = QR = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} = \sqrt{S_1}$.

6.4.1 Pair of Tangents :

From a given point $P(x_1, y_1)$ two tangents PQ and PR can be drawn to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$. Their combined equation is $SS_1 = T^2$.

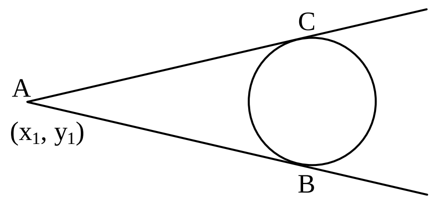
Where

$S = 0$ is the equation of circle $T = 0$ is the equation of tangent at (x_1, y_1) and S_1 is obtained by replacing x by x_1 and y by y_1 in S.



7. Chord of Contact

The chord joining the two points of contact of tangents to a circle drawn from any point A is called chord of contact of A with respect to the given circle.



Let the given point is $A(x_1, y_1)$ and the circle is S = 0 then equation of the chord of contact is

$T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

NOTE :

(i) It is clear from the above that the equation to the chord of contact coincides with the equation of the tangent, if the point (x_1, y_1) lies on the circle.

(ii) The length of chord of contact = $2\sqrt{r^2 - p^2}$

(iii) Area of Δ ABC is given by

$$\frac{a(x_1^2 + y_1^2 - a^2)^{3/2}}{x_1^2 + y_1^2}$$

8. Director Circle

The locus of the point of intersection of two perpendicular tangents to a circle is called the **Director circle**.

Let the circle be $x^2 + y^2 = a^2$, then equation of the pair of tangents to a circle from a point (x_1, y_1) is $(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$. If this represents a pair of perpendicular lines then coefficient of x^2 + coefficient of $y^2 = 0$

i.e.$(x_1^2 + y_1^2 - a^2 - x_1^2) + (x_1^2 + y_1^2 - a^2 - y_1^2) = 0$

$\Rightarrow \quad x_1^2 + y_1^2 = 2a^2$

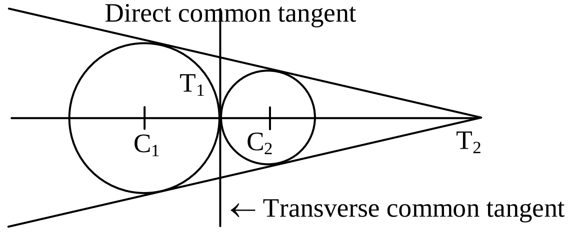
Hence the equation of director circle is $x^2 + y^2 = 2a^2$

Obviously director circle is a concentric circle whose radius is times the radius of the given circle.

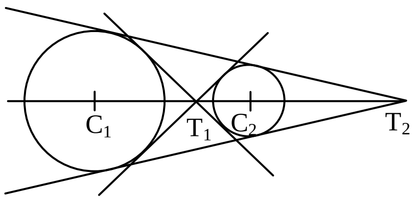
9. Position of Two Circles

Let C_1 (h_1, k_1) and C_2 (h_2, k_2) be the centre of two circle and r_1, r_2 be their radius then

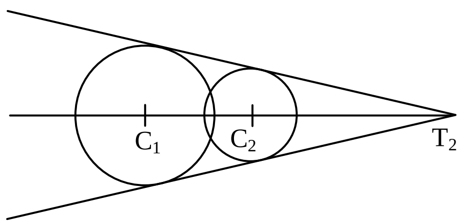
Case-I: When $C_1C_2 = r_1 + r_2$ i.e. the distance between the centres is equal to the sum of their radii. In this case, two direct tangents are real and distinct while the transverse tangents are coincident. The point T_1 divides c_1 and c_2 in the ratio of $r_1 : r_2$.



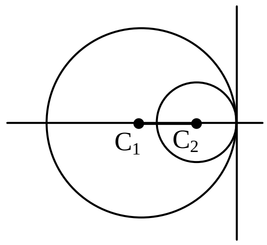
Case-II: When $C_1C_2 > r_1 + r_2$ i.e. the distance between the centres is greater than the sum of their radii. In this case , the two circles do not intersect with each other and four common tangents be drawn. Two common tangents intersects at T_2 called the direct common tangents and other two intersect at T_1 called the transverse common tangents.



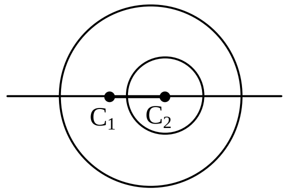
Case-III: When $(r_1 - r_2) < C_1C_2 < r_1 + r_2$ i.e. the distance between the centre is less than the sum of their radii. In this case, the two direct common tangents are real while the transverse tangents are imaginary.



Case-IV: When $C_1C_2 = | r_1 - r_2 |$ i.e. the distance between the centre is equal to the difference of their radii. In this case, two tangents are real and coincident while the other two are imaginary.



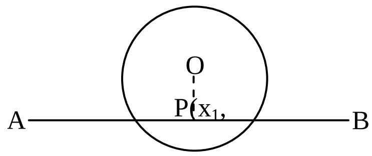
Case-V: When $C_1C_2 < |r_1 - r_2|$ i.e. the distance between centre is less than the difference of their radii. In this case, all the four common tangents are imaginary.



10. Equation of a chord whose middle point is given

The equation of the chord of the circle $x^2 + y^2 = a^2$ whose middle point $P(x_1, y_1)$ is given is

Slope of line OP $= \frac{y_1}{x_1}$; slope of AB $= \frac{x_1}{y_1}$



So equation of chord is $(y - y_1) = \frac{x_1}{y_1}(x - x_1)$

or $xx_1 + yy_1 = x_1^2 + y_1^2$.

Which can be represent by $T = S_1$

11. Circle through the Point of intersection

(i) The equation of the circle passing through the points of intersection of the circle $S = 0$ and line $L = 0$ is $S + \lambda L = 0$.

(ii) The equation of the circle passing through the points of intersection of the two circle $S = 0$ and $S' = 0$ is $S + \lambda S' = 0$. Where $(\lambda \neq -1)$

In the above both cases λ can be find out according to the given problem.

12. Common Chord of two Circles

The line joining the points of intersection of two circles is called the common chord. If the equation of two circle.

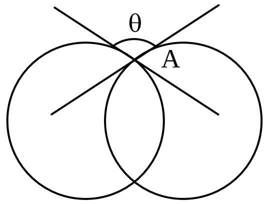
$S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and

$S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$, then equation of common chord is $S_1 - S_2 = 0$

$\Rightarrow \; 2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$

13. Angle of Intersection of two Circles

The angle of intersection between two circles $S = 0$ and $S' = 0$ is defined as the angle between their tangents at their point of intersection.



S = 0   S′ = 0

If $S \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

$S' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

are two circles with radii r_1, r_2 and d be the distance between their centres then the angle of intersection θ between them is given by

$$\cos\theta = \frac{r_1^2 + r_2^2 + d^2}{2r_1r_2}$$

or $$\cos\theta = \frac{2(g_1g_2 + f_1f_2) - (c_1 + c_2)}{2\sqrt{g_1^2 + f_1^2 - c_1}\sqrt{g_2^2 + f_2^2 - c_2}}$$

13.1 Condition of Orthogonality :

If the angle of intersection of the two circle is a right angle (θ = 90º) then such circle are called Orthogonal circle and conditions for their orthogonality is $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

SOLVED EXAMPLES

Ex.1 The lines 2x – 3y = 5 and 3x – 4y = 7 are diameters of a circle of area 154 sq. units. The equation of this circle is -

(A) $x^2 + y^2 - 2x - 2y = 47$

(B) $x^2 + y^2 - 2x - 2y = 62$

(C) $x^2 + y^2 - 2x + 2y = 47$

(D) $x^2 + y^2 - 2x + 2y = 62$

Sol. The point of intersection of the given lines is (1,–1) which is the centre of the required circle. Also if its radius be r, then as given

$\pi r^2 = 154$

$\Rightarrow r^2 = \dfrac{154 \times 7}{22} = 49 \Rightarrow r = 7$

$\therefore$ reqd. equation is $(x-1)^2 + (y+1)^2 = 49$

$\Rightarrow x^2 + y^2 - 2x + 2y = 47$ **Ans. [C]**

Ex.2 The equation of a circle which passes through the point (1,–2) and (4,–3) and whose centre lies on the line 3x + 4y = 7 is-

(A) $15(x^2 + y^2) - 94x + 18y - 55 = 0$

(B) $15(x^2 + y^2) - 94x + 18y + 55 = 0$

(C) $15(x^2 + y^2) + 94x - 18y + 55 = 0$

(D) None of these

Sol. Let the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ ...(1)

Hence, substituting the points, (1,–2) and (4,–3) in equation (1)

$$\left.\begin{aligned} 5 + 2g - 4f + c = 0 \text{ .....(2)} \\ 25 + 89 - 6f + c = 0 \text{.....(3)} \end{aligned}\right\}$$

= centre (–g,–f) lies on line 3x + 4y = 7

solving for g,f,c

Hence $-3g - 4f = 7$ ...(4)

Here $g = \dfrac{-47}{15}$, $f = \dfrac{9}{15}$, $c = \dfrac{55}{15}$

Hence the equation is

$15(x^2 + y^2) - 94x + 18y + 55 = 0$

Ans. [B]

Note: Trial method : In such cases, substitute the given points in the answer (A),(B),(C) and hence locate the correct answer. This may save time and energy.

Ex.3 The equation of a circle passing through (–4, 3) and touching the lines x+ y = 2, x– y = 2 is -

(A) $x^2 + y^2 - 20x - 55 = 0$

(B) $x^2 + y^2 + 20x + 55 = 0$

(C) $x^2 + y^2 - 20x - 55 = 0$

(D) None of these

Sol. Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

Passes through (–4, 3)

$25 - 8g + 6f + c = 0$ ...(1)

Touches both lines $\Rightarrow \dfrac{-g-f-2}{\sqrt{2}}$

$= \sqrt{g^2 + f^2 - c} = \dfrac{-g+f-2}{\sqrt{2}}$

$\therefore$ f = 0 $\therefore g^2 - 4g - 4 - 2c = 0$

Also c = 8g – 25 $\therefore g = 10 \pm 3\sqrt{6}$, f = 0, $c = 55 \pm 24\sqrt{6}$

It is easy to see that the answers given are not near to the values of g,f,c. Hence none of these is the correct option. **Ans. [D]**

Note : Correct Answer :

$x^2 + y^2 + 2(10 \pm 3\sqrt{6})x + (55 \pm 24\sqrt{6}) = 0$

Ex.4 The equation of the circle which touches the axis of y at the origin and passes through (3,4) is -

(A) $4(x^2 + y^2) - 25x = 0$

(B) $3(x^2 + y^2) - 25x = 0$

(C) $2(x^2 + y^2) - 3x = 0$

(D) $4(x^2 + y^2) - 25x + 10 = 0$

Sol. The centre of the circle lies on x– axis. Let a be the radius of the circle. Then, coordinates of the centre are (a,0). The circle passes through (3,4). Therefore,

$\sqrt{(a-3)^3 + (0-4)^2} = a$

$\Rightarrow -6a + 25 = 0 \qquad \Rightarrow a = \dfrac{25}{6}$

So, equation of the circle is

$(x - a)^2 + (y - 0)^2 = a^2$

or, $x^2 + y^2 - 2ax = 0$

or $3(x^2 + y^2) - 25x = 0$ **Ans.[B]**

Ex.5 The equation of a circle which touches x-axis and the line 4x – 3y + 4 = 0, its centre lying in the third quadrant and lies on the line $x - y - 1 = 0$, is -

(A) $9(x^2 + y^2) + 6x + 24y + 1 = 0$

(B) $9(x^2 + y^2) - 6x - 24y + 1 = 0$

(C) $9(x^2 + y^2) - 6x + 2y + 1 = 0$

(D) None of these

Sol. Let centre be $(-h,-k)$ equation

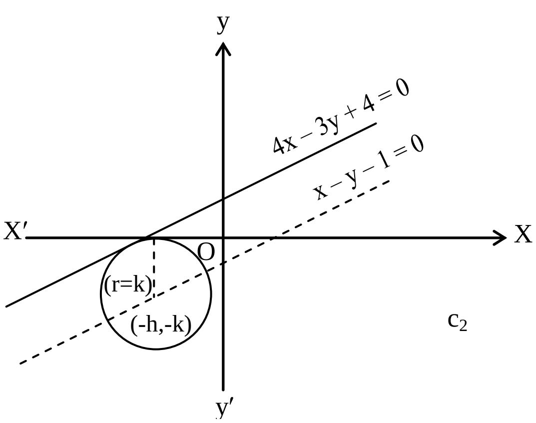
$(x + h)^2 + (y+k)^2 = k^2$ ...(1)

Also $-h + k = 1$ ...(2)

$\therefore$ $h = k-1$ radius = k (touches x- axis)

Touches the line $4x-3y + 4 = 0$

$\left|\frac{-4h - 3(-k) + 4}{5}\right| = k$ ...(3)



Solving (2) and (3), $h = \frac{1}{3}$, $k = \frac{4}{3}$

Hence the circle is

$$\left(x - \frac{4}{5}\right)^2 + \left(y + \frac{4}{3}\right)^2 = \left(\frac{4}{3}\right)^2$$

$\Rightarrow$ $9(x^2 + y^2) + 6x + 24y + 1 = 0$ **Ans.[A]**

Ex.6 The equation to a circle passing through the origin and cutting of intercepts each equal to + 5 of the axes is -

(A) $x^2 + y^2 + 5x - 5y = 0$

(B) $x^2 + y^2 - 5x + 5y = 0$

(C) $x^2 + y^2 - 5x - 5y = 0$

(D) $x^2 + y^2 + 5x + 5y = 0$

Sol. Let the circle cuts the x – axis and y– axis at A and B respectively. If O is the origin, then $\angle AOB = 90^\circ$, and A (5,0); B (0,5) is the diameter of the circle.

Then using diameter from the equation to the circle, we get

$(x - 5)(x - 0) + (y - 0)(y - 5) = 0$

$\Rightarrow$ $x^2 + y^2 - 5x - 5y = 0$ **Ans.[C]**

Ex.7 The equation of the circle whose radius is 3 and which touches the circle $x^2 + y^2 - 4x - 6y - 12 = 0$ internally at the point $(-1,-1)$ is -

(A) $(x - 4/5) + (4 + 7/5)^2 = 3^2$

(B) $(x - 4/5) + (4 - 7/5)^2 = 3^2$

(C) $(x-8)^2 + (y-1)^2 = 3^2$

(D) None of these

Sol. Let C be the centre of the given circle and C_1 be the centre of the required circle.

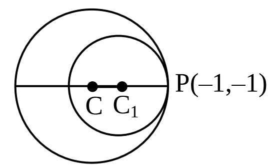
Now C = (2,3),

CP = radius = 5

$\because$ $C_1P = 3$

$\Rightarrow CC_1 = 2$

$\therefore$ The point C_1 divides internally, the line joining C and P in the ratio 2: 3



$\therefore$ coordinates of C_1 are $\left(\frac{4}{5}, \frac{7}{5}\right)$

Hence (B) is the required circle. **Ans. [B]**

Ex.8 The equation of a circle which passes through the three points (3, 0) (1, –6), (4, –1) is -

(A) $2x^2 + 2y^2 + 5x - 11y + 3 = 0$

(B) $x^2 + y^2 - 5x + 11y - 3 = 0$

(C) $x^2 + y^2 + 5x - 11y + 3 = 0$

(D) $2x^2 + 2y^2 - 5x + 11y - 3 = 0$

Sol. Let the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ ...(1)

$9 + 0 + 6g + 0 + c = 0$ ...(2)

$1 + 36 + 2g - 12f + c = 0$ ...(3)

$16 + 1 + 8g - 2f + c = 0$ ...(4)

from (2) – (3), $-28 + 4g + 12f = 0$

$g + 3f - 7 = 0$ ...(5)

from (3) – (4), $20 - 6g - 10f = 0$

$3g + 5f - 10 = 0$ ...(6)

Solving $\frac{g}{-30+35} = \frac{f}{-21+10} = \frac{1}{5-9}$

$\therefore$ $g = -\frac{5}{4}$, $f = \frac{11}{4}$, $c = -\frac{3}{2}$

Hence the circle is

$2x^2 + 2y^2 - 5x + 11y - 3 = 0$ **Ans.[D]**

Ex.9 The equation of the circle which is touched by y = x, has its centre on the positive direction of the x- axis and cuts off a chord of length 2 units along the line $\sqrt{3}\,y - x = 0$ is -

(A) $x^2 + y^2 - 4x + 2 = 0$ (B) $x^2 + y^2 - 8x + 8 = 0$
(C) $x^2 + y^2 - 4x + 1 = 0$ (D) $x^2 + y^2 - 4y + 2 = 0$

Sol. Since the required circle has its centre on X-axis, So, let the coordinates of the centre be (a,0). The circle touches y = x. Therefore,
radius = length of the perpendicular from (a,0) on $x - y = 0$

$= \frac{a}{\sqrt{2}}$

The circle cuts off a chord of length 2 units along $x - \sqrt{3}\,y = 0$.

$$\left(\frac{a}{\sqrt{2}}\right)^2 = 1^2 + \left(\frac{a - \sqrt{3}\times 0}{\sqrt{1^2 + (\sqrt{3})^2}}\right)^2$$

$$\Rightarrow \frac{a^2}{2} = 1 + \frac{a^2}{4} \Rightarrow a = 2$$

Thus, centre of the circle is at (2,0) and radius $= \frac{a}{\sqrt{2}} = \sqrt{2}$.

So, its equation is $x^2 + y^2 - 4x + 2 = 0$ **Ans.[A]**

Ex.10 The greatest distance of the point P (10, 7) from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ is -
(A) 5 (B) 15
(C) 10 (D) None of these

Sol. Since $S_1 = 10^2 + 7^2 - 4 \times 10 - 2 \times 7 - 20 > 0$. So, P lies outside the circle. Join P with the centre C (2,1) of the given circle. Suppose PC cuts the circle at A and B. Then, PB is the greatest distance of P from the circle.

We have : $PC = \sqrt{(10-2)^2 + (7-1)^2} = 10$

and CB = radius $= \sqrt{4+1+20} = 5$

$\therefore$ PB = PC + CB = (10 + 5) = 15 **Ans.[B]**

Ex.11 The length of intercept on y– axis, by a circle whose diameter is the line joining the points (–4, 3) and (12, –1) is -
(A) $2\sqrt{13}$ (B) $\sqrt{13}$
(C) $4\sqrt{13}$ (D) None of these

Sol. Here equation of the circle
$(x+4)(x-12) + (y-3)(y+1) = 0$
or $x^2 + y^2 - 8x - 2y - 51 = 0$
Hence intercept on y– axis
$= 2\sqrt{f^2 - c} = 2\sqrt{1-(-51)} = 4\sqrt{13}$ **Ans.[C]**

Ex.12 For the circle $x^2 + y^2 + 4x - 7y + 12 = 0$ the following statement is true -
(A) the length of tangent from (1, 2) is 7
(B) Intercept on y- axis is 2
(C) intercept on x– axis is $2 - \sqrt{2}$
(D) None of these

Sol. Here
(A) Putting y = 0, $x^2 + 4x + 12 = 0$ imaginary roots, not true
(B) Put x = 0, $y^2 - 7y + 12 = 0$
or $(y-3)(y-4) = 0$ intercept = 4–3 = 12
(C) Length of tangent $= \sqrt{1+4+4-14+12} = \sqrt{7}$
Hence" none of these" is true. **Ans.[D]**

Ex.13 The equation of tangent drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ is -
(A) y = 0
(B) x – y = 0
(C) $(h^2 - r^2)\,x - 2rhy = 0$
(D) None of these

Sol. Equation of tangent line drawn form origin can be taken as y = mx
The centre of the given circle is (r, h) and radius is = r.
Now by condition of tangency p = r, we have

$$\frac{mr - h}{\sqrt{1+m^2}} = \pm r$$

$$\Rightarrow m^2r^2 + h^2 - 2mhr = r^2(1+m^2)$$

$$\Rightarrow m = \frac{h^2 - r^2}{2hr}$$

Putting this value in y = mx, we get the required equation of tangent (C). **Ans.[C]**

Remark : Since we can write equation of circle in the following form $(x-r)^2 + (y-h)^2 = r^2$
Obviously, the other tangent through origin is y-axis i.e. x = 0.

Ex.14 If the squares of the lengths of the tangents from a point P to the circles $x^2 + y^2 = a^2$, $x^2 + y^2 = b^2$ and $x^2 + y^2 = c^2$ are in A.P., then
(A) a, b, c are in GP
(B) a, b, c are in AP
(C) a^2, b^2, c^2 are in AP
(D) a^2, b^2, c^2 are in GP

Sol. Let P (x_1, y_1) be the given point and PT_1, PT_2, PT_3 be the lengths of the tangents from P to the circles $x^2 + y^2 = a^2$, $x^2 + y^2 = b^2$ and $x^2 + y^2 = c^2$ respectively. Then,

$PT_1 = \sqrt{x_1^2 + y_1^2 - a^2}$, $PT_2 = \sqrt{x_1^2 + y_1^2 - b^2}$ and

$PT_3 = \sqrt{x_1^2 + y_1^2 - c^2}$

Now, PT_1^2, PT_2^2, PT_3^2 are in AP

$\Rightarrow$ $2\,PT_2^2 = PT_1^2 + PT_3^2$

$\Rightarrow$ $2(x_1^2 + y_1^2 - b^2) = (x_1^2 + y_1^2 - a^2) + (x_1^2 + y_1^2 - c^2)$

$\Rightarrow$ $2b^2 = a^2 + c^2$

$\Rightarrow$ a^2, b^2, c^2 are in AP. **Ans.[C]**

Ex.15 The area of the triangle formed by the tangents from an external point (h, k) to the circle $x^2 + y^2 = a^2$ and the chord of contact, is -

(A) $\frac{1}{2}a\left(\frac{h^2 + k^2 - a^2}{\sqrt{h^2 + k^2}}\right)$

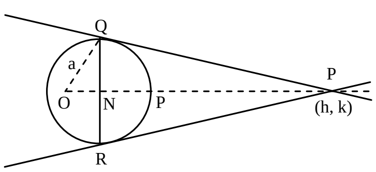
(B) $\frac{a(h^2 + k^2 - a^2)^{3/2}}{2(h^2 + k^2)}$

(C) $\frac{a(h^2 + k^2 - a^2)^{3/2}}{(h^2 + k^2)}$

(D) None of these

Sol. Here area of Δ PQR is required

Now chord of contact w.r. to circle $x^2 + y^2 = a^2$, and point (h, k) $hx + ky - a^2 = 0$



Perp. from (h, k), PN = $\frac{h^2 + k^2 - a^2}{\sqrt{h^2 + k^2}}$

Also length QR = $2\sqrt{a^2 - \frac{(a^2)^2}{h^2 - k^2}}$

$= \frac{2a\sqrt{h^2 + k^2 - a^2}}{\sqrt{h^2 + k^2}}$

$\therefore$ $\Delta PQR = \frac{1}{2}$ (QR) (PN)

$= \frac{1}{2}2a\sqrt{\frac{h^2 + k^2 - a^2}{h^2 + k^2}}\ \frac{(h^2 + k^2 - a^2)}{\sqrt{h^2 + k^2}}$

$= a\,\frac{(h^2 + k^2 - a^2)^{3/2}}{\sqrt{h^2 + k^2}}$ **Ans.[C]**

Ex.16 If the line y = x + 3 meets the circle $x^2 + y^2 = a^2$ at A and B, then the equation of the circle having AB as a diameter will be -

(A) $x^2 + y^2 + 3x - 3y - a^2 + 9 = 0$

(B) $x^2 + y^2 + 3x + 3y - a^2 + 9 = 0$

(C) $x^2 + y^2 - 3x + 3y - a^2 + 9 = 0$

(D) None of these

Sol. Let the equation of the required circle be

$(x^2 + y^2 - a^2) + \lambda\,(y - x - 3) = 0$

since its centre $(\lambda/2, -\lambda/2)$ lies on the given line, so we have $-\lambda/2 = \lambda/2 + 3 = -3$

Putting this value of in (A) we get the reqd. eqn. as

$x^2 + y^2 + 3x - 3y - a^2 + 9 = 0$ **Ans. [A]**

Ex.17 The equation of the circle passing through the point of intersection of the circles $x^2 + y^2 = 6$ and $x^2 + y^2 - 6x + 8 = 0$, and also through the point (1, 1) is -

(A) $x^2 + y^2 - 4y + 2 = 0$

(B) $x^2 + y^2 - 3x + 1 = 0$

(C) $x^2 + y^2 - 6x + 4 = 0$

(D) None of these

Sol. Let the equation of the required circle be

$(x^2 + y^2 - 6x + 8) + (x^2 + y^2 - 6) = 0$

Since it passes through (1, 1), so we have

$1 + 1 - 6 + 8 + \lambda\,(1 + 1 - 6) = 0 = 1$

$\therefore$ the required equation is

$x^2 + y^2 - 3x + 1 = 0$ **Ans. [B]**

Ex.18 If y = 2x is a chord of the circle $x^2 + y^2 = 10x$, then the equation of the circle whose diameter is this chord is -

(A) $x^2 + y^2 + 2x + 4y = 0$

(B) $x^2 + y^2 + 2x - 4y = 0$

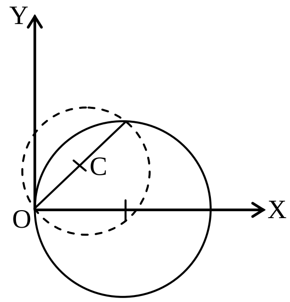
(C) $x^2 + y^2 - 2x - 4y = 0$

(D) None of these

Sol. Here equation of the circle

$(x^2 + y^2 - 10x) + \lambda(y - 2x) = 0$

Now centre C $(5 + \lambda, -\lambda/2)$ lies on the



chord again

$\therefore \frac{-\lambda}{2} = 2(5+\lambda) \Rightarrow \frac{-5\lambda}{2} = 10$

$\therefore \lambda = -4$

Hence $x^2 + y^2 = 10x - 4y + 8x = 0$

or $x^2 + y^2 - 2x - 4y = 0$ **Ans.[C]**

Ex.19 The circle S_1 with centre C_1 (a_1, b_1) and radius r_1 touches externally the circle S_2 with centre C_2 (a_2, b_2) and radius r_2. If the tangent at their common point passes through the origin, then

(A) $(a_1^2 + a_2^2) + (b_1^2 + b_2^2) = r_1^2 + r_2^2$

(B) $(a_1^2 - a_2^2) + (b_1^2 - b_2^2) = r_2^2 - r_1^2$

(C) $(a_1^2 - b_2^2) + (a_2^2 + b_2^2) = r_1^2 + r_2^2$

(D) $(a_1^2 - b_1^2) + (a_2^2 + b_2^2) = r_1^2 + r_2^2$

Sol. The two circles are

$S_1 = (x - a_1)^2 + (y - b_1^2) = r_1^2$ ...(i)

$S_2 = (x - a_2)^2 + (y - b_2^2) = r_2^2$ ...(ii)

The equation of the common tangent of these two circles is given by $S_1 - S_2 = 0$

i.e., $2x(a_1 - a_2) + 2y(b_1 - b_2) + (a_2^2 + b_2^2)$

$- (a_1^2 + b_1^2) + r_1^2 - r_2^2 = 0$

If this passes through the origin, then

$(a_2^2 + b_2^2) - (a_1^2 + b_1^2) + r_1^2 - r_2^2 = 0$

$(a^2{}_2 - a_1^2) + (b_2^2 - b_1^2) = r^2{}_2 - r_1^2$ **Ans.[B]**

Ex.20 The length of the common chord of the circles $(x-a)^2 + y^2 = c^2$ and $x^2 + (y-b)^2 = c^2$ is -

(A) $\sqrt{c^2 + a^2 + b^2}$ (B) $\sqrt{4c^2 + a^2 + b^2}$

(C) $\sqrt{4c^2 - a^2 - b^2}$ (D) $\sqrt{c^2 - a^2 - b^2}$

Sol. The equation of the common chord is

$[(x-a)^2 + y^2 - c^2] - [x^2 + (y-b)^2 - c^2] = 0$

$\Rightarrow 2ax - 2by - a^2 + b^2 = 0$ ...(1)

Now p = length of perpendicular from (a, 0) on (1)

$= \frac{2a^2 - a^2 + b^2}{\sqrt{4a^2 + 4b^2}} = \frac{1}{2}\sqrt{a^2 + b^2}$

$\therefore$ length of common chord

$= 2\sqrt{c^2 - p^2} = 2\sqrt{c^2 - \frac{a^2 + b^2}{4}}$

$= \sqrt{4c^2 - a^2 - b^2}$ **Ans.[C]**

Ex.21 The angle of intersection of the two circles $x^2 + y^2 - 2x - 2y = 0$ and $x^2 + y^2 = 4$, is -

(A) 30º (B) 60º (C) 90º (D) 45º

Sol. Here circles are

$x^2 + y^2 - 2x - 2y = 0$ ...(1)

$x^2 + y^2 = 4$ ...(2)

Now c_1 (1, 1), $r_1 = \sqrt{1^2 + 1^2} = \sqrt{2}$

c_2 (0, 0), $r_2 = 2$

If θ is the angle of intersection then

$\cos\theta = \frac{r_1^2 + r_2^2 - (c_1c_2)^2}{2r_1r_2}$

$= \frac{2 + 4 - (\sqrt{2})^2}{2.\sqrt{2}.2.} = \frac{1}{\sqrt{2}}$

$= \theta = 45^\circ$ **Ans.[D]**

Ex.22 If a circle passes through the point (1,2) and cuts the circle $x^2 + y^2 = 4$ orthogonally, then the locus of its centre is -

(A) $x^2 + y^2 - 2x - 6y - 7 = 0$

(B) $x^2 + y^2 - 3x - 8y + 1 = 0$

(C) $2x + 4y - 9 = 0$

(D) $2x + 4y - 1 = 0$

Sol. Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0,$

Since it passes through (1, 2), so

$1 + 4 + 2g + 4f + c = 0$

$\Rightarrow 2g + 4f + c + 5 = 0$ ...(1)

Also this circle cuts $x^2 + y^2 = 4$ orthogonally, so $2g(0) + 2f(0) = c - 4$

$\Rightarrow c = 4$ ...(2)

From (1) and (2) eliminating c, we have

$2g + 4f + 9 = 0$

Hence locus of the centre (–g, –f) is

$2x + 4y - 9 = 0$ **Ans.[C]**

Ex.23 Circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 2x - 4y + 3 = 0$

(A) touch each other externally

(B) touch each other internally

(C) intersect each other

(D) do not intersect

Sol. Here C_1 (0, 0) and C_2 (1, 2)

$\therefore C_1 C_2 = \sqrt{1+4} = \sqrt{5} = 2.23.$

Also $r_1 = 2$, $r_2 = \sqrt{1+4-3} = \sqrt{2} = 1.41$

$\therefore r_1 - r_2 < C_1C_2 < r_1 + r_2$

$\Rightarrow$ circles intersect each other. **Ans.[C]**

Ex.24 The circles $x^2 + y^2 + 2x - 2y + 1 = 0$ and $x^2 + y^2 - 2x - 2y + 1 = 0$ touch each other -

(A) externally at (0,1) (B) internally at (0,1)

(C) externally at (1,0) (D) internally at (1,0)

Sol. The centres of the two circles are C_1 (–1, 1) and C_2 (1, 1) and both have radii equal to 1. We have:

$C_1C_2 = 2$ and sum of the radii = 2

So, the two circles touch each other externally. The equation of the common tangent is obtained by subtracting the two equations.

The equation of the common tangent is

$4x = 0 \Rightarrow x = 0.$

Putting x = 0 in the equation of the either circle, we get

$y^2 - 2y + 1 = 0 \Rightarrow (y-1)^2 = 0 \Rightarrow y = 1.$

Hence, the points where the two circles touch is (0,1). **Ans.[A]**

Ex.25 The total number of common tangents to the two circles $x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$, is -

(A) 1 (B) 2 (C) 3 (D) 4

Sol. Here

c_1 (1, 3), $r_1 = \sqrt{1+9-9} = 1$

c_2 (–3, 1), $r_2 = \sqrt{9+1-1} = 3$

Now $c_1c_2 = \sqrt{(1+3)^2 + (3-2)^2}$

$= \sqrt{16+1} = \sqrt{17}$

$c_1c_2 > r_1 + r_2$

Hence the circles are non- intersecting externally. Hence 4 tangents, two direct and two transverse tangents may be drawn. **Ans.[D]**

Ex.26 If (4, –2) is a point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, which is concentric to $x^2 + y^2 - 2x + 4y + 20 = 0$, then value of c is -

(A) – 4 (B) 0

(C) 4 (D) 1

Sol. Since the first circle is concentric to $x^2 + y^2 - 2x + 4y + 20 = 0$, therefore its equation can be written as

$x^2 + y^2 - 2x + 4y + c = 0$

If it passes through (4,–2), then

$16 + 4 - 8 - 8 + c = 0$

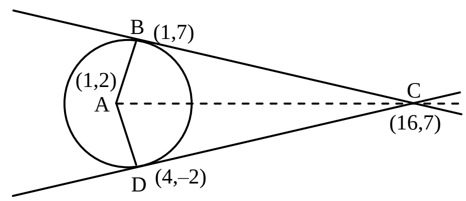
$\Rightarrow c = -4$ **Ans. [A]**

Ex.27 Let A be the centre of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$, and B(1,7) and D(4,–2) are points on the circle then, if tangents be drawn at B and D, which meet at C, a then area of quadrilateral ABCD is -

(A) 150 (B) 75

(C) 75/2 (D) None of these

Sol.



Here centre A (1,2), and Tangent at (1,7) is

$x.1 + y.7 - 1(x+1) - 2(y+7) - 20 = 0$

or $y = 7$ ...(1)

Tangent at D (4,–2) is

$3x - 4y - 20 = 0$ ...(2)

Solving (1) and (2), C is (16, 7)

Area ABCD = AB x BC

$= 5 \times \sqrt{256+49-32-28-20}$

$= 5 \times 15 = 75$ units **Ans.[B]**

Ex.28 The abscissa of two points A and B are the roots of the equation $x^2 + 2ax - b^2 = 0$ and their ordinates are the roots of the equation $y^2 + 2py - q^2 = 0$. The radius of the circle with AB as a diameter will be -

(A) $\sqrt{a^2+b^2+p^2+q^2}$ (B) $\sqrt{b^2+q^2}$

(C) $\sqrt{a^2+b^2-p^2-q^2}$ (D) $\sqrt{a^2+p^2}$

Sol. Let $A \equiv (\alpha, \beta)$; $B \equiv (\gamma, \delta)$. Then

$\alpha + \gamma = -2a$, $\alpha\gamma = -b^2$

and $\beta + \delta = -2p$, $\beta\delta = -q^2$

Now equation of the required circle is

$(x - \alpha)(x - \gamma) + (y - \beta)(y - \delta) = 0$

$\Rightarrow x^2 + y^2 - (\alpha + \gamma)x - (\beta + \delta) + \alpha\gamma + \beta\delta = 0$

$\Rightarrow x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0$

Its radius $= \sqrt{a^2 + b^2 + p^2 + q^2}$ **Ans.[A]**

Ex.29 Two rods of length a and b slide on the axes in such a way that their ends are always concyclic. The locus of centre of the circle passing through the ends is -

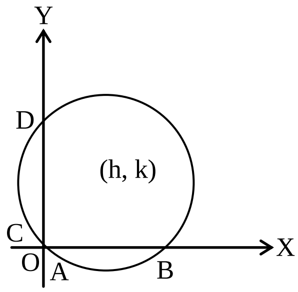
(A) $4(x^2 - y^2) = a^2 - b^2$

(B) $x^2 - y^2 = a^2 - b^2$

(C) $x^2 - y^2 = 4(a^2 - b^2)$

(D) $x^2 + y^2 = a^2 + b^2$

Sol. Let a rod AB of length 'a' slides on x-axis and rod CD of length 'b' slide on y - axis so that ends A, B, C and D are always concyclic.



Let equation of circle passing through these ends is

$x^2 + y^2 + 2gx + 2fy + c = 0$

Obviously $2\sqrt{g^2 - c} = a$ and $2\sqrt{f^2 - c} = b$

$\therefore\ 4(g^2 - f^2) = a^2 - b^2$

$\Rightarrow 4[(-g)^2 - (-f)^2] = a^2 - b^2$

therefore locus of centre $(-g, -f)$ is

$4(x^2 - y^2) = a^2 - b^2$. **Ans.[A]**

Ex.30 The angle between the tangents from α, β to the circle $x^2 + y^2 = a^2$ is -

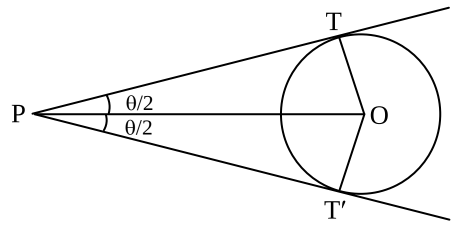
(A) $\tan^{-1}\left(\frac{a}{\sqrt{S_1}}\right)$ (B) $2\tan^{-1}\left(\frac{a}{\sqrt{S_1}}\right)$

(C) $2\tan^{-1}\left(\frac{\sqrt{S_1}}{a}\right)$ (D) None of these

Where $S_1 = \alpha^2 + \beta^2 - a^2$

Sol. Let PT and PT' be the tangents drawn from P (α, β) to the circle $x^2 + y^2 = a^2$, and let $\angle TPT' = \theta$. If O is the centre of the circle, then $\angle TPO = \angle T'PO = \theta/2$.

$\therefore \tan\frac{\theta}{2} = \frac{OT}{OP} = \frac{a}{\sqrt{S_1}}$



$\Rightarrow \frac{\theta}{2} = \tan^{-1}\left(\frac{a}{\sqrt{S_1}}\right)$

$\Rightarrow \theta = 2\tan^{-1}\left(\frac{a}{\sqrt{S_1}}\right)$ **Ans. [B]**