COMPLEX NUMBER

(KEY CONCEPTS + SOLVED EXAMPLES)

COMPLEX NUMBER

1. *The Real number system*
2. *Imaginary number*
3. *Complex number*
4. *Modulus of a complex number*
5. *Amplitude of a complex number*
6. *Square root of a complex number*
7. *Triangle inequalities*
8. *Miscellaneous results*

KEY CONCEPTS

1. The Real Number System

Natural Number (N) : The number which are used for counting are known as Natural Number (also known as set of Positive Integers) i.e.

N = {1, 2, 3, ...}

Whole Number (W) : If ' 0 ' is included in the set of natural numbers then we get the set of Whole Numbers i.e.W = {0, 1, 2, ...}

= {N} + {0}

Integers (Z or I) :If negative natural number is included in the set of whole number then we get set of Integers i.e.

Z or I = {.............–3, –2, –1, 0, 1, 2, 3,.............}

Rational Numbers (Q) : The numbers which are in the form of p/q (Where p, q ∈ I, q ≠ 0) are called as Rational Number e.g. $\sqrt{2}$, $\frac{2}{3}$, 3, $\frac{1}{3}$, 0.76, 1.2322 etc.

Irrational Numbers : The numbers which are not rational i.e. which can not be expressed in p/q form or whose decimal part is non terminating non repeating but which may represent magnitude of physical quantities. e.g., $5^{1/3}$, π, e,.....etc.

Real Numbers (R) : The set of Rational and Irrational Number is called as set of Real Numbers i.e. $N \subset W \subset Z \subset Q \subset R$

Note :

(i) Number zero is neither positive nor negative but is an even number.

(ii) Square of a real number is always positive.

(iii) Between two real numbers there lie infinite real numbers.

(iv) The real number system is totally ordered, for any two numbers a, b ∈R , we must say, either a < b or b < a or b = a.

(v) All real number can be represented by points on a straight line. This line is called as number line.

(vi) An Integer (Note) is said to be even, if it is divided by 2 other wise it is odd number.

(vii) The magnitude of a physical quantity may be expressed as a real number times, a standard unit.

(viii) Number ' 0 ' is an additive quantity

(ix) Number '1' is multiplicative quantity.

(x) Infinity (∞) is the concept of the number greater than greatest you can imagine. It is not a number, it is just a concept, so we do not associate equality with it.

(xi) Division by zero is meaning less.

(xii) A non zero integer p is called prime if p ≠ ± 1 and its only divisors are ± 1 and ± p.

1.1 Modulus of a Real Number :

The Modulus of a real number x is defined as follows

$|x| = x$ when $x > 0$

0 when $x = 0$

$-x$ when $x < 0$

e.g. $|3| = 3$ $\quad |-6| = -(-6) = 6$

Now $|x-a| = \begin{cases} x-a & \text{when } x \geq a \\ -(x-a) & \text{when } x < a \end{cases}$

1.2 Intervals : Let a, x, b are real number so that

$x \in [a, b] \Rightarrow a \leq x \leq b$

[a,b] is known as the closed interval a, b

$x \in (a, b) \Rightarrow a < x < b$

(a, b) is known as the open interval a, b

$x \in (a, b] \Rightarrow a < x \leq b$

(a, b] is known as semi open, semi closed Interval

$x \in [a, b) \Rightarrow a \leq x < b$

[a, b) is known as semi closed, semi open Interval

2. Imaginary Number

Square root of a negative real number is an imaginary number, while solving equation $x^2 + 1 = 0$ we get $x = \pm\sqrt{-1}$ which is imaginary. So the quantity $\sqrt{-1}$ is denoted by 'i' called 'iota' thus $i = \sqrt{-1}$

Further $\sqrt{-2}$, $\sqrt{-3}$, $\sqrt{-4}$ ....may be expressed as $\pm i\sqrt{2}$, $\pm i\sqrt{3}$, $\pm 2i$ ....

2.1 Integral powers of iota

As we have seen $i = \sqrt{-1}$ so $i^2 = -1$

$i^3 = -i$ and $i^4 = 1$

Hence $n \in N$, $i^n = i, -1, -i, 1$ attains four values according to the value of n, so

$$i^{4n+1} = i, \qquad i^{4n+2} = -1$$

$$i^{4n+3} = -i, \qquad i^{4n} \text{ or } i^{4n+4} = 1$$

In other words $i^n = (-1)^{n/2}$ if n is even integer

$$i^n = (-1)^{\frac{n-1}{2}}\, i \quad \text{if n is odd integer.}$$

Note :–

(i) $i^2 = i \times i = \sqrt{-1} \times \sqrt{-1} \neq \sqrt{1}$

(ii) $\sqrt{a.b} = \sqrt{a}\,.\sqrt{b}$ possible iff both a, b are non-negative. (incorrect). It is also true for one positive and one negative no.

e.g. $\sqrt{(-2)(3)} = \sqrt{-2}\,.\sqrt{3}$

only invalid when both are negative means

$\sqrt{a.b} \neq \sqrt{a}\,.\sqrt{b}$ iff a & b both are negative.

(iii) ' i ' is neither positive, zero nor negative, Due to this reason order relations are not defined for imaginary numbers.

3. Complex Number

A number of the form $z = x + iy$ where $x, y \in R$ and $i = \sqrt{-1}$ is called a complex number where x is called as real part and y is called imaginary part of complex number and they are expressed as

$Re(z) = x$, $Im(z) = y$

Here if $x = 0$ the complex number is purely Imaginary and if $y = 0$ the complex number is purely Real.

A complex number may also be defined as an ordered pair of real numbers any may be denoted by the symbol (a, b). If we write z = (a, b) then a is called the real part and b the imaginary part of the complex number z.

Note :

(i) Inequalities in complex number are not defined because 'i' is neither positive, zero nor negative so $4 + 3i < 1 + 2i$ or $i < 0$

or $i > 0$ is meaning less.

(ii) If two complex numbers are equal, then their real and imaginary parts are separately equal. Thus if $a + ib = c + id$

$\Rightarrow a = c$ and $b = d$

so if $z = 0 \Rightarrow x + iy = 0 \Rightarrow x = 0$ and $y = 0$

The student must note that

$x, y \in R$ and $x, y \neq 0$. Then if

$x + y = 0 \Rightarrow x = y$ is correct

but $x + iy = 0 \Rightarrow x = -iy$ is incorrect

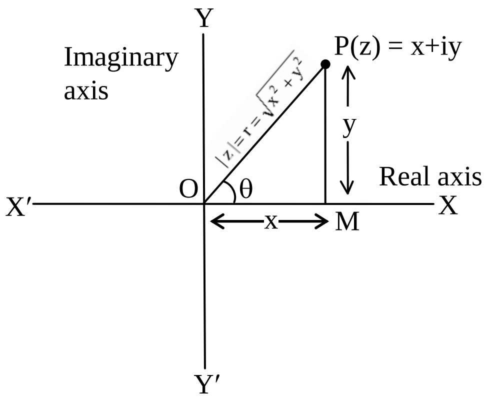
Hence a real number cannot be equal to the imaginary number, unless both are zero.

(iii) The complex number 0 is purely real and purely imaginary both.

3.1 Representation of a Complex Number :

(a) Cartesian Representation :

The complex number $z = x + iy = (x, y)$ is represented by a point P whose coordinates are refered to rectangular axis xox´ and yoy´, which are called real and imaginary axes respectively. Thus a complex number z is represented by a point in a plane, and corresponding to every point in this plane there exists a complex number such a plane is called Argand plane or Argand diagram or complex plane or gussian plane.



Note :

(i) Distance of any complex number from the origin is called the modulus of complex number and is denoted by |z|. Thus, $|z| = \sqrt{x^2 + y^2}$.

(ii) Angle of any complex number with positive direction of x-axis is called amplitude or argument of z. Thus, amp (z) = arg (z) = θ $= \tan^{-1}\frac{y}{x}$.

(b) Polar Representation : If z = x + iy is a complex number then z = r (cos θ + i sin θ) is a polar form of complex number z where x = r cos θ, y = r sin θ and $r = \sqrt{x^2 + y^2} = |z|$.

(c) Exponential Form : If z = x + iy is a complex number then its exponential form is $z = r e^{i\theta}$ where r is modulas and θ is amplitude of complex number.

(d) Vector Representation : If z = x + iy is a complex number such that it represent point P(x, y) then its vector representation is $z = \overrightarrow{OP}$

3.2 Algebraic operations with Complex Number:

Addition (a + ib) + (c + id) = (a + c) + i (b + d)

Subtraction (a + ib)–(c + id) = (a – c) + i (b – d)

Multiplication $(a + ib)(c + id) = ac + iad + ibc + i^2bd$
$= (ac – bd) + i(ad + bc)$

Division $\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)}$

(when at least one of c and d is non zero)

$$= \frac{(ac+bd)}{c^2+d^2} + i\frac{(bc-ad)}{c^2+d^2}$$

3.2.1 Properties of Algebraic operations with Complex Number

Let z, z_1, z_2 and z_3 are any complex number then their algebraic operation satisfy following properties-

Commutativity : $z_1 + z_2 = z_2 + z_1$ & $z_1 z_2 = z_2 z_1$

Associativity : $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

and $(z_1 z_2) z_3 = z_1(z_2 z_3)$

Identity element : If O = (0, 0) and 1 = (1, 0) then z + 0 = 0 + z = z and z.1 = 1. z = z. Thus 0 and 1 are the identity elements for addition and multiplication respectively.

Inverse element : Additive inverse of z is – z and multiplicative inverse of z is $\frac{1}{z}$.

Cancellation Law :

$$\left.\begin{matrix} z_1 + z_2 = z_1 + z_3 \\ z_2 + z_1 = z_3 + z_1 \end{matrix}\right\} \Rightarrow z_2 = z_3$$

and $z_1 \neq 0$ $\left.\begin{matrix} z_1 z_2 = z_1 z_3 \\ z_2 z_1 = z_3 z_1 \end{matrix}\right\} \Rightarrow z_2 = z_3$

Distributivity : $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

and $(z_2 + z_3) z_1 = z_2 z_1 + z_3 z_1$

3.3 Conjugate Complex Number :

The complex numbers z = (a, b) = a + ib and $\bar{z}$ = (a, – b) = a – ib where b ≠ 0 are said to be complex conjugate of each other (Here the complex conjugate is obtained by just changing the sign of i) e.g.conjugate of z = – 3 + 4i is $\bar{z}$ = – 3 – 4i.

Note : Image of any complex number in x-axis is called its conjugate.

3.3.1 Properties of Conjugate Complex Number

Let z = a + ib and $\bar{z}$ = a – ib then

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} = 2a = 2\,\text{Re}(z)$ = purely real

(iii) $z - \bar{z} = 2ib = 2i\,\text{Im}(z)$ = purely imaginary

(iv) $z\bar{z} = a^2 + b^2 = |z|^2$

(v) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(vi) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(vii) $\overline{re^{i\theta}} = re^{-i\theta}$

(viii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}$

(ix) $\overline{z^n} = (\overline{z})^n$

(x) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$

(xi) $z + \overline{z} = 0$ or $z = -\overline{z}$

$\Rightarrow$ z = 0 or z is purely imaginary

(xii) $z = \overline{z} \Rightarrow$ z is purely real

4. Modulus of a Complex Number

If z = x + iy then modulus of z is equal to $\sqrt{x^2 + y^2}$ and it is denoted by |z|. Thus

$$z = x + iy \Rightarrow |z| = \sqrt{x^2 + y^2}$$

Note :

Modulus of every complex number is a non negative real number.

4.1 Properties of modulus of a Complex Number

(i) $|z| \geq 0$

(ii) $-|z| \leq \text{Re}(z) \leq |z|$

(iii) $-|z| \leq \text{Im}(z) \leq |z|$

(iv) $|z| = |\overline{z}| = |-z| = |-\overline{z}|$

(v) $z\,\overline{z} = |z|^2$

(vi) $|z_1 z_2| = |z_1|\ |z_2|$

(vii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ $(z_2 \neq 0)$

(viii) $|z|^n = |z^n|$, $n \in N$

(ix) $|z| = 1 \Leftrightarrow \overline{z} = \frac{1}{z}$

(x) $z^{-1} = \frac{\overline{z}}{|z|^2}$

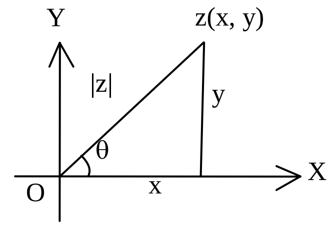
(xi) $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\text{Re}(z_1 \overline{z}_2)$

(xii) $|z_1+z_2|^2 + |z_1-z_2|^2 = 2[|z_1|^2 + |z_2|^2]$

(xiii) $|re^{i\theta}| = r$

5. Amplitude or Argument of a Complex Number

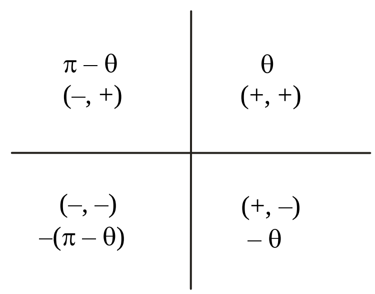
The amplitude or argument of a complex number z is the inclination of the directed line segment representing z, with real axis.



If $z = x + iy$ then

$$\text{amp}(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

For finding the argument of any complex number first check that the complex number is in which quadrant and then find the angle θ and amplitude using the adjacent figure.



Note :

(i) Principle value of any complex number lies between $-\pi < \theta \leq \pi$.

(ii) Amplitude of a complex number is a many valued function. If θ is the argument of a complex number then $(2n\pi+\theta)$ is also argument of complex number.

(iii) Argument of zero is not defined.

(iv) If a complex number is multiplied by iota (i) its amplitude will be increased by π/2 and will be decreased by π/2, if is multiplied by –i.

(v) Amplitude of complex number in I and II quadrant is always positive and in III and IV is always negative.

5.1 Properties of argument of a Complex Number

(i) amp (any real positive number) = 0

(ii) amp (any real negative number) = π

(iii) amp $(z - \bar{z}) = \pm \pi/2$

(iv) amp $(z_1 . z_2)$ = amp (z_1) + amp (z_2)

(v) amp $\left(\frac{z_1}{z_2}\right)$ = amp (z_1) – amp (z_2)

(vi) amp $(\bar{z})$ = – amp (z) = amp (1/z)

(vii) amp (– z) = amp (z) $\pm \pi$

(viii) amp (z^n) = n amp (z)

(ix) amp (iy) = $\pi/2$ if y > 0

= – $\pi/2$, if y < 0

(x) amp (z) + amp $(\bar{z})$ = 0

6. Square root of a Complex Number

The square root of z = a + ib is -

$$\sqrt{a+ib} = \pm\left[\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}}\right] \text{ for } b > 0$$

$$\text{and } \pm\left[\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|-a}{2}}\right] \text{ for } b < 0$$

Note :

(i) The square root of i is $\pm\left(\frac{1+i}{\sqrt{2}}\right)$ (Here b = 1)

(ii) The square root of – i is $\pm\left(\frac{1-i}{\sqrt{2}}\right)$ (Here b = –1)

(iii) The square root of ω is $\pm \omega^2$

(iv) The square root of ω^2 is $\pm \omega$

7. Triangle Inequalities

(i) $|z_1 \pm z_2| \le |z_1| + |z_2|$

(ii) $|z_1 \pm z_2| \ge |z_1| - |z_2|$

8. Miscellaneous Results

(i) If ABC is an equilateral triangle having vertices z_1, z_2, z_3 then $z_1^2 + z_2^2 + z_3^2$

$= z_1z_2 + z_2z_3 + z_3z_1$

or $\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0.$

(ii) If z_1, z_2, z_3, z_4 are vertices of parallelogram then $z_1 + z_3 = z_2 + z_4$.

(iii) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers represented by points P and Q respectively in Argand Plane then -

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= |(x_2 - x_1) + i(y_2 - y_1)| = |z_2 - z_1|$$

(iv) If a point P divides AB in the ratio of m : n, then $z = \frac{mz_2 + nz_1}{m+n}$ where z_1, z_2 and z represents the point A, B and P respectively.

(v) $|z - z_1| = |z - z_2|$ represents a perpendicular bisector of the line segment joining the points z_1 and z_2.

(vi) Let P be any point on a circle whose centre C and radius r, let the affixes of P and C be z and z_0 then $|z - z_0| = r$.

(a) Again if $|z - z_0| < r$ represent interior of the circle of radius r.

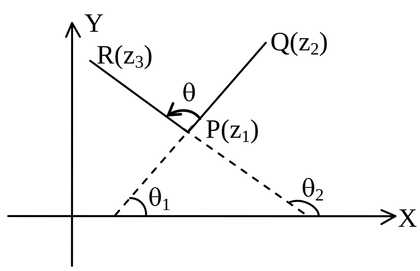
(b) $|z - z_0| > r$ represent exterior of the circle of radius r.

(vii) Let z_1, z_2, z_3 be the affixes of P, Q, R respectively in the Argand Plane. Then from the figure the angle between PQ and PR is.

$\theta = \theta_2 - \theta_1$

$= \arg.\ \overrightarrow{PR} - \arg \overrightarrow{PQ}$

$= \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$



(a) If z_1, z_2, z_3 are collinear, thus $\theta = 0$ therefore $\frac{z_3 - z_1}{z_2 - z_1}$ is purely real.

(b) If z_1, z_2, z_3 are such that PR $\perp$ PQ,

$\theta = \pi / 2$ So $\frac{z_3 - z_1}{z_2 - z_1}$ is purely imaginary.

SOLVED EXAMPLE

Ex.1 $\sqrt{-2}\ \sqrt{-3}$ is equal to -

(A) $i\sqrt{6}$ (B) $-\sqrt{6}$

(C) $\sqrt{6}$ (D) None of these

Sol. $\sqrt{-2} \times \sqrt{-3} = \sqrt{2}i \times \sqrt{3}i$

$= \sqrt{6}\ (i)^2 = -\sqrt{6}$ **Ans.[B]**

Ex.2 If x be real, the relation in a and b, when $\frac{1-ix}{1+ix} = a - ib$, is

(A) ab = 1 (B) $a^2 - b^2 = 1$

(C) $a^2 + b^2 = 1$ (D) None of these

Sol. $\because \frac{1-ix}{1+ix} = a - ib$

on taking modulus; we get

$|a - ib| = \left|\frac{1-ix}{1+ix}\right|$

$\Rightarrow \sqrt{a^2+b^2} = \left|\frac{1-ix}{1+ix}\right| = \frac{|1-ix|}{|1+ix|} = 1$

$\therefore a^2 + b^2 = 1$ **Ans.[3]**

Ex.3 If the vertices of any quadrilateral are A = 1+ 2i, B = –3 + i, C = –2 – 3i , and D = 2 – 2i, then it is-

(A) Parallelogram (B) Rhombus

(C) Square (D) Rectangle

Sol. A = (1, 2), B = (–3, 1)

C = (–2,–3), D = (2, –2)

$\therefore AB = \sqrt{(-3-1)^2+(1-2)^2} = \sqrt{17}$

$BC = \sqrt{(-2+3)^2+(-3-1)^2} = \sqrt{17}$

$CD = \sqrt{(2+2)^2+(-2+3)^2} = \sqrt{17}$

$DA = \sqrt{(1-2)^2+(2+2)^2} = \sqrt{17}$

Diagonal $AC = \sqrt{(-2-1)^2+(-3-2)^2} = \sqrt{34}$

and $BD = \sqrt{(2+3)^2+(-2-1)^2} = \sqrt{34}$

$\because$ AB= BC = CD = DA and AC = BD

$\therefore$ ABCD is a square **Ans.[3]**

Ex.4 If $z = \left(\frac{1}{2}, 1\right)$, then the value of z^{-1} is-

(A) $\left(-\frac{2}{5}, \frac{4}{5}\right)$ (B) $\left(\frac{1}{5}, -\frac{2}{5}\right)$

(C) $\left(\frac{1}{5}, \frac{2}{5}\right)$ (D) $\left(\frac{2}{5}, -\frac{4}{5}\right)$

Sol. $z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{(1/2)-i}{(1/2)^2+1} = \frac{2}{5} - \frac{4}{5}i$

$= \left(\frac{2}{5}, -\frac{4}{5}\right)$ **Ans.[D]**

Ex.5 If $\frac{\tan\theta - i\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)}{1+2i\sin\frac{\theta}{2}}$ is purely imaginary then general value of θ is-

(A) $n\pi + \frac{\pi}{4}$ (B) $2n\pi + \frac{\pi}{4}$

(C) $n\pi + \frac{\pi}{2}$ (D) $2n\pi + \frac{\pi}{2}$

Sol. Multiply above and below by conjugate of denominator and put real part equal to zero.

$= \frac{\tan\theta - i\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)}{1+2i\sin\frac{\theta}{2}} \times \frac{1-2i\sin\frac{\theta}{2}}{1-2i\sin\frac{\theta}{2}}$

$= \frac{\tan\theta - 2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right) - i\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}+2\tan\theta\sin\frac{\theta}{2}\right)}{1+4\sin^2\frac{\theta}{2}}$

$\therefore \tan\theta - 2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right) = 0$

$\Rightarrow \frac{\sin\theta}{\cos\theta} - (1-\cos\theta) - \sin\theta = 0$

$\Rightarrow \sin\theta\left(\frac{1-\cos\theta}{\cos\theta}\right) - (1-\cos\theta) = 0$

$\Rightarrow (1-\cos\theta)(\tan\theta - 1) = 0$

$\cos\theta = 1 \Rightarrow \theta = 2n\pi$ and

$\tan\theta = 1 \Rightarrow \theta = n\pi + \frac{\pi}{4}$ **Ans.[A]**

Ex.6 For any two non real complex numbers z_1, z_2 if $z_1 + z_2$ and z_1z_2 are real numbers, then

(A) $z_1 = 1/z_2$ (B) $z_1 = \bar{z}_2$

(C) $z_1 = -z_2$ (D) $z_1 = z_2$

Sol. Let $z_1 = a + ib$ and $z_2 = c + id$ ($b \neq 0$, $d \neq 0$).

Then $z_1 + z_2$ and z_1z_2 are real

$\Rightarrow$ $b + d = 0$ and $ad + bc = 0$

$\Rightarrow$ $d = -b$ and $c = a (\because b \neq 0, d \neq 0)$

$\Rightarrow$ $z_1 = \bar{z}_2$ **Ans.[B]**

Ex.7 In a complex plane z_1, z_2, z_3, z_4 taken in order are vertices of parallelogram if

(A) $z_1 + z_2 = z_3 + z_4$ (B) $z_1 + z_3 = z_2 + z_4$

(C) $z_1 + z_4 = z_2 + z_3$ (D) None of these

Sol. Let the given points be A, B, C, D respectively. Then ABCD is a parallelogram, so-

$\overrightarrow{AB} = \overrightarrow{DC}$

$\Rightarrow$ $z_2 - z_1 = z_3 - z_4$

$\Rightarrow$ $z_1 + z_3 = z_2 + z_4$ **Ans.[B]**

Ex.8 The complex numbers sin x + i cos 2x and cos x – i sin 2x are conjugate to each other when-

(A) $x = 0$ (B) $x = \left(n + \frac{1}{2}\right)\pi$

(C) $x = n\pi$ (D) no value of x

Sol. $\sin x + i \cos 2x = \cos x + i \sin 2x$

$\Rightarrow$ $\tan x = 1$ and $\tan 2x = 1$

$\Rightarrow$ $x = n\pi + \frac{\pi}{4}$ and $x = \frac{n\pi}{2} + \frac{\pi}{8}$

$\Rightarrow$ $x \in \left\{....,\frac{-7\pi}{4},\frac{-3\pi}{4},\frac{\pi}{4},\frac{5\pi}{4},\frac{9\pi}{4},......\right\}$

$\cap \left\{....,\frac{-7\pi}{8},\frac{-3\pi}{8},\frac{\pi}{8},\frac{5\pi}{8},\frac{9\pi}{8},......\right\}$

$\Rightarrow$ there is no common value of x.

Ans. [D]

Ex.9 If A, B and C are respectively the complex numbers 3 + 4i, 5–2i, –1+ 16i, then A, B, C are-

(A) collinear

(B) vertices of right-angle triangle

(C) vertices of isosceles triangle

(D) vertices of equilateral triangle

Sol. Given points are A(3,4), B(5,–2) and C(– 1, 16).

Now slope of AB = $\frac{-2-4}{5-3} = -3$

slope of BC = $\frac{16+2}{-1-5} = -3$

$\therefore$ slope of AB = slope of BC

$\Rightarrow$ A, B, C are collinear. **Ans.[A]**

Ex.10 If complex numbers z_1, z_2 and 0 are vertices of an equilateral triangle, then $z_1^2 + z_2^2 - z_1z_2$ is equal to-

(A) 0 (B) $z_1 - z_2$

(C) $z_1 + z_2$ (D) 1

Sol. z_1, z_2, 0 are vertices of an equilateral triangle, so we have

$z_1^2 + z_2^2 + 0^2 = z_1z_2 + z_2.0 + 0.z_1$

(a property)

$\Rightarrow$ $z_1^2 + z_2^2 = z_1z_2$

$\Rightarrow$ $z_1^2 + z_2^2 - z_1z_2 = 0$. **Ans. [A]**

Ex.11 If $w = \frac{z-(1/5)i}{z}$ and $|w| = 1$, then complex number z lies on

(A) a parabola (B) a circle

(C) a line (D) None of these

Sol. $|w| = 1 \Rightarrow |z - (1/5)i| = |z|$

$\Rightarrow$ $|z - (1/5)i|^2 = |z|^2$

$\Rightarrow$ $|x + iy - 1/5i|^2 = |x + iy|^2$

$\Rightarrow$ $x^2 + (y - 1/5)^2 = x^2 + y^2$

$\Rightarrow$ $-2/5y + 1/25 = 0$

$\Rightarrow$ $10y = 1$, which is a line. **Ans.[C]**

Ex.12 If complex numbers z_1, z_2, z_3 represent the vertices of an equilateral triangle such that $|z_1| = |z_2| = |z_3|$;then-

(A) $I(z_1 + z_2 + z_3) = 0$ (B) $z_1 + z_2 + z_3 = 0$

(C) $R(z_1 + z_2 + z_3) = 0$ (D) None of these

Sol. Let A, B, C denote complex numbers z_1, z_2, z_3.

Then $|z_1| = |z_2| = |z_3| \Rightarrow OA = OB = OC$

$\Rightarrow$ O is the circumcentre of Δ ABC

$\Rightarrow$ O is the centroid of Δ ABC

($\because$ it is equilateral)

$\Rightarrow$ $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \vec{0}$

$\Rightarrow$ $z_1 + z_2 + z_3 = 0$ **Ans.[B]**

Ex.13 If z_1, z_2 are any two complex numbers and a, b are any two real numbers, then $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$ is equal to-

(A) $(a^2 + b^2)(|z_1|^2 + |z_2|^2)$

(B) $a^2b^2(|z_1|^2 + |z_2|^2)$

(C) $(a + b)^2(|z_1|^2 + |z_2|^2)$

(D) None of these

Sol. Expression

$= (az_1 - bz_2)\,\overline{(az_1 - bz_2)}$

$\quad + (bz_1 + az_2)\,\overline{(bz_1 + az_2)}$

$= (az_1 - bz_2)(a\bar{z}_1 - b\bar{z}_2)$

$\quad + (bz_1 + az_2)(b\bar{z}_1 + a\bar{z}_2)$

$= a^2|z_1|^2 + b^2|z_2|^2 + b^2|z_1|^2 + a^2|z_2|^2$

$= (a^2 + b^2)(|z_1|^2 + |z_2|^2)$ **Ans.[A]**

Ex.14 If $z = x + iy$, and if $\log_{\sqrt{3}} \frac{|z|^2 - |z| + 1}{2 + |z|} < 2$ then z lies in the interior of the circle

(A) $|z| = 4$ (B) $|z| = 3$

(C) $|z| = 2$ (D) $|z| = 5$

Sol. $\log_{\sqrt{3}} \frac{|z|^2 - |z| + 1}{2 + |z|} < 2$

$\Rightarrow \frac{|z|^2 - |z| + 1}{2 + |z|} < (\sqrt{3})^2$

$\Rightarrow |z|^2 - |z| + 1 < 6 + 3|z|$

$\Rightarrow |z|^2 - 4|z| - 5 < 0$

$\Rightarrow (|z| - 5)(|z| + 1) \Rightarrow (|z| - 5) < 0$

since $|z| + 1 > 0 \Rightarrow |z| < 5$

Hence z lies inside the circle $|z| = 5$

Ans.[D]

Ex.15 The amplitude of $1 - \cos\theta - i\sin\theta$ is-

(A) $\frac{1}{2}(\pi - \theta)$ (B) $\frac{\theta}{2}$

(C) $-\frac{\pi}{2} + \frac{\theta}{2}$ (D) $\frac{\pi}{2} + \frac{\theta}{2}$

Sol. Let

$z = 1 - \cos\theta - i\sin\theta = r(\cos\phi + i\sin\phi)$

$\therefore \quad \tan\phi = -\frac{\sin\theta}{1 - \cos\theta}$

$= \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}$

$= -\cot(\theta/2)$

$= -\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$

or $\quad \tan\phi = \tan\left(\frac{\theta}{2} - \frac{\pi}{2}\right)$

$\therefore \quad \text{amp}(z) = \frac{\theta}{2} - \frac{\pi}{2}$ **Ans.[C]**

Ex.16 If $x_n = \cos(\pi/2^n) + i\sin(\pi/2^n)$, then $x_1x_2x_3 \ldots\ldots\ldots\infty$ is equal to-

(A) -1 (B) 1

(C) 0 (D) ∞

Sol. $x_1x_2x_3 \ldots\ldots\infty$

$= \cos\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \ldots\right)$

$+ i\sin\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \ldots\right)$ **Ans.[A]**

Ex.17 If $z_1 = 10 + 6i$, $z_2 = 4 + 6i$ and z is a complex number such that $\text{amp}\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{4}$, then $|z - 7 - 9i|$ is equal to-

(A) $2\sqrt{2}$ (B) $\sqrt{2}$

(C) $3\sqrt{2}$ (D) $2\sqrt{3}$

Sol. If $z = x + iy$, then $\text{amp}\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{4}$

$\Rightarrow x^2 + y^2 - 14x - 18y + 112 = 0 \quad \ldots(1)$

Now $|z - 7 - 9i|$

$= \sqrt{x^2 + y^2 - 14x - 18y + 130}$

$= 3\sqrt{2}$ (from 1) **Ans.[C]**

Ex.18 The polar form of complex number

$z = \frac{\{\cos(\pi/3) - i\sin(\pi/3)\}(\sqrt{3} + i)}{i - 1}$ is-

(A) $\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$

(B) $\sqrt{2}\left(\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right)$

(C) $\sqrt{2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$

(D) None of these

Sol. Here |z|

$$= \frac{|\cos(\pi/3) - i\sin(\pi/3)||\sqrt{3}+i|}{|i-1|} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Again amp(z) = amp {cos(π /3) – i sin (π /3)} + amp ($\sqrt{3}$ + i) – amp (– 1 + i)

$$= -\frac{\pi}{3} + \frac{\pi}{6} - \left(\pi - \frac{\pi}{4}\right) = -\frac{11\pi}{12}$$

Therefore

$$z = \sqrt{2}\left\{\cos\left(-\frac{11\pi}{12}\right) + i\sin\left(-\frac{11\pi}{12}\right)\right\}$$

$$= \sqrt{2}\left\{\cos\left(-\frac{11\pi}{12} + 2\pi\right) + i\sin\left(-\frac{11\pi}{12} + 2\pi\right)\right\}$$

$$= \sqrt{2}\left\{\cos\left(\frac{13\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right)\right\}$$ **Ans.[B]**

Ex.19 If $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$ then $\left(\frac{z_1}{z_2}\right)$ is

(A) zero or purely imaginary
(B) purely imaginary
(C) purely real
(D) None of these

Sol. ∵ $|z_1 + z_2|^2 = |z_1|^2\ |z_2|^2 + 2|z_1||z_2| \cos(\theta_1 - \theta_2)$

∴ If $\theta_1 - \theta_2 = \pm\frac{\pi}{2}$;

Then $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$

i.e. $\text{Arg}(z_1) - \text{Arg}(z_2) = \pm\frac{\pi}{2}$

$\Rightarrow \text{Arg}\left(\frac{z_1}{z_2}\right) = \pm\frac{\pi}{2}$

$\Rightarrow \frac{z_1}{z_2}$ is purely imaginary **Ans.[B]**

Ex.20 Square root of – 8 – 6i is -

(A) ± (3 + i) (B) ± (1 + i$\sqrt{3}$)
(C) ± (1 – 3i) (D) ± (1 + 3i)

Sol. Let $\sqrt{-8-6i} = \pm(a + ib)$

⇒ $-8 - 6i = a^2 - b^2 + 2iab$

⇒ $a^2 - b^2 = -8$ ...[1]

$2ab = -6 \Rightarrow ab = -3$ ...[2]

$(a^2 + b^2)^2 = (a^2 - b^2)^2 + 4a^2b^2$

$= (-8)^2 + (-6)^2$

$= 64 + 36 = 100$

⇒ $a^2 + b^2 = 10$ ...[3]

from equation (2) and (3)

a = 1, b = –3

So, $\sqrt{-8-6i} = \pm(1 - 3i)$ **Ans.[C]**

Ex.21 If z = x + iy, $z^{1/3}$ = a – ib and $\frac{x}{a} - \frac{y}{b} = k(a^2 - b^2)$, then k equals-

(A) –2 (B) 2 (C) 4 (D) 0

Sol. Here $x + iy = (a - ib)^3$

$= (a^3 - 3ab^2) + i(-3a^2b + b^3)$

⇒ $x = a^3 - 3ab^2$, $y = b^3 - 3a^2b$

⇒ $\frac{x}{a} - \frac{y}{b} = (a^2 - 3b^2) - (b^2 - 3a^2)$

$= 4(a^2 - b^2)$

⇒ k = 4 **Ans.[C]**

Ex.22 The complex number z having least positive argument which satisfy the condition

|z – 25i| ≤ 15 is -

(A) 25i (B) 12 + 25i
(C) 16 + 12i (D) 12 + 16i

Sol. The required complex number is point of contact P as shown in the figure. C(0, 25) is the centre of the circle and radius is 15.

Now |z| = OP

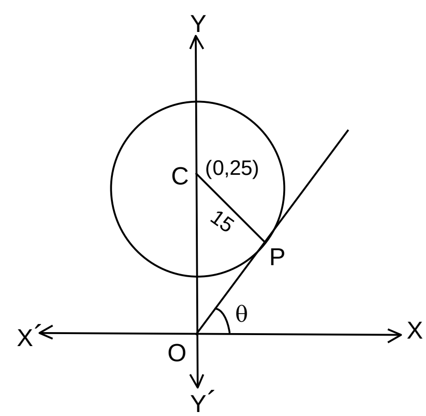
$= \sqrt{OC^2 - PC^2}$

$= \sqrt{625 - 225} = 20$

amp (z) = θ = ∠ XOP = ∠ OCP

∴ $\cos\theta = \frac{PC}{OC} = \frac{15}{25} = \frac{3}{5}$

and $\sin\theta = \frac{OP}{OC} = \frac{20}{25} = \frac{4}{5}$



$\therefore z = 20\left(\frac{3}{5} + \frac{4}{5}i\right)$

= 12 + 16i. **Ans.[D]**

Ex.23 If $|z + 2i| \le 1$, then greatest and least value of $|z-\sqrt{3} + i|$ are-

(A) 3, 1 (B) ∞, 0

(C) 1, 3 (D) None of these

Sol. $|z - \sqrt{3} + i| = |(z + 2i) - (\sqrt{3} + i)|$

$\le |(z + 2i)| + |(\sqrt{3} + i)|$

$\le 1 + 2 = 3$

$\Rightarrow$ The greatest value of $|z-\sqrt{3} + i|$ is 3.

Again $|z - \sqrt{3} + i|$

$= |(z + 2i) - (\sqrt{3} + i)|$

$\ge |\sqrt{3} + i| - |z + 2i|$

$\ge 2 - 1 = 1$

Thus least value of $|z - \sqrt{3} + i|$ is 1. **Ans.[A]**

Ex.24 The value of $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$ is -

(A) – i (B) 0

(C) – 1 (D) i

Sol. $\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$

$= -i\left(\cos\frac{2\pi k}{7} + i\sin\frac{2\pi k}{7}\right) = -i\,e^{\frac{2\pi k i}{7}}$

$\therefore \sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$

$= -i\left[e^{\frac{2\pi i}{7}} + e^{\frac{4\pi i}{7}} + \ldots\ldots 6 \text{ terms}\right]$

$= -i\,e^{\frac{2\pi i}{7}}\left\{\frac{1 - e^{\frac{12\pi i}{7}}}{1 - e^{\frac{2\pi i}{7}}}\right\}$ $(\because e^{2\pi i} = 1)$

$= -i\left\{\frac{e^{\frac{2\pi i}{7}} - 1}{1 - e^{\frac{2\pi i}{7}}}\right\} = i$ **Ans.[D]**

Ex.25 If z_0 is the circumcenter of an equilateral triangle with vertices z_1, z_2, z_3, then $z_1^2 + z_2^2 + z_3^2$ is equal to-

(A) z_0^2 (B) $2\,\frac{z_0^2}{3}$

(C) $3\,z_0^2$ (D) $\frac{z_0^2}{3}$

Sol. Since z_1, z_2, z_3, are vertices of an equilateral triangle, so

$z_1^2 + z_2^2 + z_3^2$

$= z_1 z_2 + z_2z_3 + z_3z_1$ ...(1)

Further the circumcenter of an equilateral triangle is same as its centroid, so

$z_0 = (z_1 + z_2 + z_3)/3$

$\Rightarrow 9z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1z_2 + z_2z_3 + z_3z_1)$

$= z_1^2 + z_2^2 + z_3^2 + 2(z_1^2 + z_2^2 + z_3^2)$

$\therefore z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.

Ans.[C]