ELLIPSE

(KEY CONCEPTS & SOLUTIONS)

══ELLIPSE══

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KEY CONCEPTS

1. Definition

An ellipse is the locus of a point which moves in such a way that its distance form a fixed point is in constant ratio to its distance from a fixed line. The fixed point is called the **focus** and fixed line is called the **directrix** and the constant ratio is called the **eccentricity of a ellipse** denoted by (e).

In other word, we can say an ellipse is the locus of a point which moves in a plane so that the sum of it distances from fixed points is constant.

2. Equation of an Ellipse

2.1 Standard Form of the equation of ellipse

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1} \quad (a > b)$$

Let the distance between two fixed points S and S' be 2ae and let C be the mid point of SS′.

Taking CS as x- axis, C as origin.

Let P(h, k) be the moving point Let SP+ SP′ = 2a (fixed distance) then

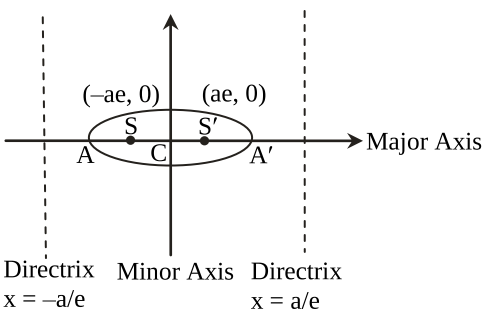
$SP+S'P = \sqrt{\{(h-ae)^2+k^2\}} + \sqrt{\{(h+ae)^2+k^2\}} = 2a$

$h^2(1-e^2) + k^2 = a^2(1-e^2)$

Hence Locus of P(h, k) is given by.

$x^2(1-e^2) + y^2 = a^2(1-e^2)$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$



Let us assume that $a^2(1-e^2) = b^2$

∴ The standard equation will be given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

2.1.1 Various parameter related with standard ellipse :

Let the equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a > b)

(i) **Vertices of an ellipse :**

The points of the ellipse where it meets with the line joining its two foci are called its vertices.

For above standard ellipse A. A′ are vertices

$A \equiv (a, 0), A' \equiv (-a, 0)$

(ii) **Major axis :**

The chord AA′ joining two vertices of the ellipse is called its major axis.

Equation of major axis : y = 0

Length of major axis = 2a

(iii) **Minor axis :**

The chord BB′ which bisects major axis AA′ perpendicularly is called minor axis of the ellipse.

Equation of minor axis x = 0

Length of minor axis = 2b

(iv) **Centre :**

The point of intersection of major axis and minor axis of an ellipse is called its centre.

For above standard ellipse

centre = C(0, 0)

(v) **Directrix :**

Equation of directrices are x= a/e and x = – a/e.

(vi) **Focus :** S (ae, 0) and S′ (– ae, 0) are two foci of an ellipse.

(vii) **Latus Rectum :** Such chord which passes through either focus and perpendicular to the major axis is called its latus rectum.

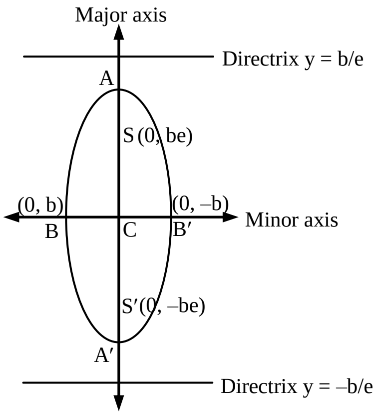
(viii) **Length of Latus Rectum :**

Length of Latus rectum is given by $\frac{2b^2}{a}$.

(ix) **Relation between constant a, b, and e**

$b^2 = a^2(1-e^2) \Rightarrow e = \sqrt{1-\frac{b^2}{a^2}}$

3. Second form of Ellipse



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{when } a < b.$$

For this ellipse

(i) centre : (0, 0)

(ii) vertices : (0, b) ; (0, – b)

(iii) foci : (0, be) ; (0, – be)

(iv) major axis : equation x = 0, length = 2b

(v) minor axis : equation y = 0, length = 2a

(vi) directrices : y = b/e, y = – b/e

(vii) length of latus ractum = $2a^2/b$

(viii) eccentricity : $e = \sqrt{1-\frac{a^2}{b^2}}$

4. General equation of the ellipse

The general equation of an ellipse whose focus is (h,k) and the directrix is the line ax + by + c = 0 and the eccentricity will be e. Then let $P(x_1,y_1)$ be any point on the ellipse which moves such that SP = ePM

$$\Rightarrow (x_1-h)^2+(y_1-k)^2=\frac{e^2(ax_1+by_1+c)^2}{a^2+b^2}$$

Hence the locus of (x_1,y_1) will be given by

$(a^2+b^2)\,[(x-h)^2+(y-k)^2]=e^2(ax+by+c)^2$

Which is the equation of second degree from which we can say that any equation of second degree represent equation of an ellipse.

Note : Condition for second degree in X & Y to represent an ellipse is that if $h^2 = ab < 0$ & $\Delta = abc + 2\,fgh - af^2 - bg^2 - ch^2 \neq 0$

5. Parametric forms of the Ellipse

Let the equation of ellipse in standard form will be given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Then the equation of ellipse in the parametric form will be given by $x = a\cos\phi$, $y = b\sin\phi$ where ϕ is the eccentric angle whose value vary from $0 \le \phi < 2\pi$. Therefore coordinate of any point P on the ellipse will be given by $(a\cos\phi, b\sin\phi)$.

6. Point and Ellipse

Let $P(x_1, y_1)$ be any point and let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the equation of an ellipse.

The point lies outside, on or inside the ellipse as if

$$S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>0,\ =0,\ <0$$

7. Ellipse and a Line

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the given line be $y = mx + c$.

Solving the line and ellipse we get

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$$

i.e. $(a^2m^2+b^2)\,x^2+2\,mca^2\,x+a^2\,(c^2-b^2)=0$

above equation being a quadratic in x.

$\therefore$ discriminant $= 4m^2c^2a^4 - 4a^2\,(a^2m^2+b^2)\,(c^2-b^2)$

$= b^2\,\{(a^2m^2+b^2)-c^2\}$

Hence the line intersects the ellipse in

(i) two distinct points if $a^2m^2+b^2>c^2$

(ii) in one point if $c^2=a^2m^2+b^2$

(iii) does not intersect if $a^2m^2+b^2<c^2$

$\therefore\ y = mx \pm\sqrt{(a^2m^2+b^2)}$ touches the ellipse and condition for tangency $c^2=a^2m^2+b^2$.

Hence $y = mx \pm\sqrt{(a^2m^2+b^2)}$, touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $\left(\frac{\pm a^2m}{\sqrt{a^2m^2+b^2}},\frac{\pm b^2}{\sqrt{a^2m^2+b^2}}\right)$.

8. Equation of the Tangent

(i) The equation of the tangent at any point (x_1, y_1) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

(ii) The equation of tangent at any point ‘ϕ’ is

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1.$$

SOLVED EXAMPLES

Ex.1 The equation of an ellipse whose focus is (–1, 1), eccentricity is 1/2 and the directrix is $x - y + 3 = 0$ is.

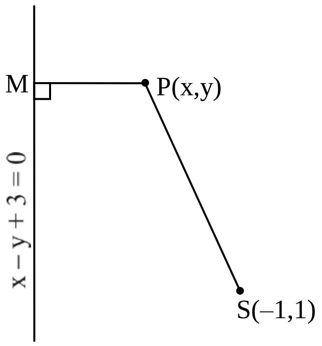
(A) $7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0$

(B) $7x^2 + 7y^2 + 2xy - 10x - 10y + 7 = 0$

(C) $7x^2 + 7y^2 + 2xy - 10x + 10y + 7 = 0$

(D) None of these

Sol.[A] Let P (x,y) be any point on the ellipse whose focus is S (–1,1) and the directrix is $x - y + 3 = 0$.



PM is perpendicular from P (x,y) on the directrix

$x - y + 3 = 0.$

Then by definition

SP = ePM

$\Rightarrow (SP)^2 = e^2 (PM)^2$

$\Rightarrow (x + 1)^2 + (y - 1)^2 = \frac{1}{4}\left\{\frac{x-y+3}{\sqrt{2}}\right\}^2$

$\Rightarrow 8 (x^2 + y^2 + 2x - 2y + 2)$

$= x^2 + y^2 + 9 - 2xy + 6x - 6y$

$\Rightarrow 7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0$

which is the required equation of the ellipse.

Ex.2 The foci of an ellipse are (±2, 0) and its eccentricity is 1/2, the equation of ellipse is.

(A) $\frac{x^2}{16}+\frac{y^2}{9}=1$ (B) $\frac{x^2}{16}+\frac{y^2}{12}=1$

(C) $\frac{x^2}{4}+\frac{y^2}{2}=1$ (D) None of these

Sol.[B] Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

Then coordinates of foci are (± ae, 0).

$\therefore$ ae = 2 $\Rightarrow a \times \frac{1}{2} = 2$ $\left[\because e = \frac{1}{2}\right]$

$\Rightarrow a = 4$

We have $b^2 = a^2 (1 - e^2)$

$\therefore b^2 = 16\left(1-\frac{1}{4}\right) = 12$

Thus, the equation of the ellipse is $\frac{x^2}{16} + \frac{y^2}{12} = 1$.

Ex.3 The equation of the ellipse which passes through origin and has its foci at the points (1, 0) and (3, 0) is -

(A) $3x^2 + 4y^2 = x$ (B) $3x^2 + y^2 = 12x$

(C) $x^2 + 4y^2 = 12x$ (D) $3x^2 + 4y^2 = 12x$

Sol.[D] Centre being mid point of the foci is

$\left(\frac{1+3}{2}, 0\right) = (2, 0)$

Distance between foci 2ae = 2

ae = 1 or $b^2 = a^2 (1 - e^2)$

$b^2 = a^2 - a^2e^2 \Rightarrow a^2 - b^2 = 1$ …(i)

If the ellipse $\frac{(x-2)^2}{a^2} + \frac{y^2}{b^2} = 1$, then as it passes from (0, 0)

$\frac{4}{a^2} = 1 \Rightarrow a^2 = 4$

from (i) $b^2 = 3$

Hence $\frac{(x-2)^2}{4} + \frac{y^2}{3} = 1$

or $3x^2 + 4y^2 - 12x = 0$

Ex.4 A man running round a racecourse notes that the sum of the distance of two flag posts from him is always 10 meters and the distance between the flag posts is 8 meters. The area of the path he encloses -

(A) 10π (B) 15π

(C) 5π (D) 20π

Sol.[B] The race course will be an ellipse with the flag posts as its foci. If a and b are the semi major and minor axes of the ellipse, then sum of focal distances 2a = 10 and 2ae = 8

a = 5, e = 4/5

$\therefore b^2 = a^2(1 - e^2) = 25\left(1-\frac{16}{25}\right) = 9$

Area of the ellipse = πab

$= \pi.5.3 = 15\pi$

Ex.5 The distance of a point on the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$ from the centre is 2. Then eccentric angle of the point is -

(A) $\pm\frac{\pi}{2}$ (B) $\pm\pi$

(C) $\frac{\pi}{4},\frac{3\pi}{4}$ (D) $\pm\frac{\pi}{4}$

Sol.[C] Any point on the ellipse is

$(\sqrt{6}\cos\phi, \sqrt{2}\sin\phi)$, where ϕ is an eccentric angle.

It's distance from the center (0, 0) is given 2.

$6\cos^2\phi + 2\sin^2\phi = 4$

or $3\cos^2\phi + \sin^2\phi = 2$

$2\cos^2\phi = 1$

$\Rightarrow \cos\phi = \pm\frac{1}{\sqrt{2}}$; $\phi = \frac{\pi}{4},\frac{3\pi}{4}$

Ex.6 The equation of tangents to the ellipse $9x^2 + 16y^2 = 144$ which pass through the point (2, 3) -

(A) $y = 3$ (B) $x + y = 2$

(C) $x - y = 3$ (D) $y = 3$; $x + y = 5$

Sol.[D] Ellipse $9x^2 + 16y^2 = 144$

or $\frac{x^2}{16}+\frac{y^2}{9}=1$

Any tangent is $y = mx + \sqrt{16m^2+9}$ it passes through (2, 3)

$3 = 2m + \sqrt{16m^2+9}$

$(3 - 2m)^2 = 16m^2 + 9$

$m = 0, -1$

Hence the tangents are $y = 3$, $x + y = 5$

Ex.7 The line $x = at^2$ meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the real points if -

(A) $|t| < 2$ (B) $|t| \le 1$

(C) $|t| > 1$ (D) None of these

Sol.[B] Putting $x = at^2$ in the equation of the ellipse, we get

$\frac{a^2t^4}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y^2 = b^2(1 - t^4)$

$y^2 = b^2(1 - t^2)(1 + t^2)$

This will give real values of y if

$(1 - t^2) \ge 0$ $|t| \le 1$

Ex.8 The equation $x^2 + 4y^2 + 2x + 16y + 13 = 0$ represents a ellipse -

(A) whose eccentricity is $\sqrt{3}$

(B) whose focus is $(\pm\sqrt{3}, 0)$

(C) whose directrix is $x = \pm\frac{4}{\sqrt{3}} - 1$

(D) None of these

Sol.[C] We have $x^2 + 4y^2 + 2x + 16y + 13 = 0$

$(x^2 + 2x + 1) + 4(y^2 + 4y + 4) = 4$

$(x + 1)^2 + 4(y + 2)^2 = 4$

$\frac{(x+1)^2}{2^2}+\frac{(y+2)^2}{1^2}=1$

Comparing with $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$

where $X = x + 1$, $Y = y + 2$

and $a = 2$, $b = 1$

eccentricity of the ellipse

$e = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$

Focus of the ellipse $(\pm ae, 0)$

$X = \pm ae$ and $Y = 0$

$x + 1 = \pm 2.\frac{\sqrt{3}}{2}$ and $y + 2 = 0$

$\Rightarrow x = -1 \pm \sqrt{3}$ and $y = -2$

$\therefore$ Focus $(-1 \pm \sqrt{3}, -2)$

Directrix of the ellipse $X = \pm a/e$

$x + 1 = \pm\frac{2}{\sqrt{3}/2}$; $x = \pm\frac{4}{\sqrt{3}} - 1$

Ex.9 Product of the perpendiculars from the foci upon any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is -

(A) b (B) a

(C) a^2 (D) b^2

Sol.[D] The equation of any tangent to the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y = mx + \sqrt{a^2m^2+b^2}$

$\Rightarrow mx - y + \sqrt{a^2m^2+b^2} = 0$ ...(i)

The two foci of the given ellipse are S(ae, 0) and S′ (–ae, 0). let p_1 and p_2 be the lengths of perpendicular from S and S′ respectively on (i), Then

p_1 = length of perpendicular from S(ae, 0) on (i)

$$p_1 = \frac{mae + \sqrt{a^2m^2 + b^2}}{\sqrt{m^2+1}}$$

p_2 = length of perpendicular from S′(–ae, 0) on (i)

$$p_2 = \frac{-mae + \sqrt{a^2m^2 + b^2}}{\sqrt{m^2+1}}$$

Now p_1p_2

$$\left(\frac{mae + \sqrt{a^2m^2 + b^2}}{\sqrt{m^2+1}}\right)\left(\frac{-mae + \sqrt{a^2m^2 + b^2}}{\sqrt{m^2+1}}\right)$$

$$= \frac{a^2m^2(1-e^2)+b^2}{1+m^2} \quad \because b^2 = a^2(1-e^2)$$

$$= \frac{m^2b^2+b^2}{1+m^2} = \frac{b^2(m^2+1)}{m^2+1} = b^2$$

Ex.10 The equation of the ellipse whose axes are along the coordinate axes, vertices are (± 5,0) and foci at (± 4,0) is.

(A) $\frac{x^2}{25}+\frac{y^2}{9}=1$ (B) $\frac{x^2}{25}+\frac{y^2}{16}=1$

(C) $\frac{x^2}{25}+\frac{y^2}{12}=1$ (D) None of these

Sol.[A] Let the equation of the required ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad ...(1)$$

The coordinates of its vertices and foci are (± a, 0) and (± ae,0) respectively.

a = 5 and ae = 4 ⇒ e = 4/5.

Now, $b^2 = a^2(1-e^2) \Rightarrow b^2 = 25\left(1-\frac{16}{25}\right) = 9$.

Substituting the values of a^2 and b^2 in (1), we get

$$\frac{x^2}{25}+\frac{y^2}{9} = 1,$$

which is the equation of the required ellipse.

Ex.11 Find the centre, the length of the axes and the eccentricity of the ellipse $2x^2+3y^2-4x-12y+13=0$.

Sol. The given equation can be rewritten as $2[x^2-2x]+3[y^2-4y]+13=0$

or $2(x-1)^2+3(y-2)^2=1$

or $\frac{(x-1)^2}{(1/\sqrt{2})^2} + \frac{(y-2)^2}{(1/\sqrt{3})^2} = 1,$

Comparing with $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$

∴ Centre X = 0, Y = 0 i.e. (1,2)

Length of major axis = 2a = $\sqrt{2}$

Length of minor axis = 2b = $2/\sqrt{3}$ and

$$e = \sqrt{1-\frac{b^2}{a^2}} = \frac{1}{\sqrt{3}}$$

Ex.12 Find the equations of the tangents to the ellipse $4x^2 + 3y^2 = 5$ which are inclined at an angle of 60° to the axis of x. Also, find the point of contact.

Sol. The slope of the tangent = tan 60° = $\sqrt{3}$

Now, $4x^2+3y^2=5 \Rightarrow \frac{x^2}{5/4}+\frac{y^2}{5/3}=1$

This is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where

$a^2=\frac{5}{4}$ and $b^2=\frac{5}{3}$. We know that the equations of the tangents of slope m to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are given by $y = mx \pm \sqrt{a^2m^2+b^2}$ and the coordinates of the points of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$

Here, m = $\sqrt{3}$, $a^2 = 5/4$ and $b^2 = 5/3$.

So, the equations of the tangents are

$$y = \sqrt{3}\,x \pm \sqrt{\left(\frac{5}{4}\times 3\right)+\frac{5}{3}} \quad \text{i.e. } y = \sqrt{3}\,x \pm \sqrt{\frac{65}{12}}$$

The coordinates of the points of contact are $\left(\pm\frac{5\sqrt{3}/4}{\sqrt{65/12}}, \mp\frac{5/3}{\sqrt{65/12}}\right)$ i.e .

$$\left(\pm\frac{3\sqrt{65}}{26}, \mp\frac{2\sqrt{195}}{39}\right)$$

Ex.13 The radius of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, and having its centre (0, 3) is -

(A) 4 (B) 3

(C) $\sqrt{12}$ (D) 7/2

Sol.[A] $e = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{9}{16}}$ $\therefore e = \frac{\sqrt{7}}{4}$

$\therefore$ Foci are $(\pm ae, 0)$ or $(\pm\sqrt{7}, 0)$

Centre of circle is (0, 3) and passes through foci $(\pm\sqrt{7}, 0)$

$\therefore$ Radius $= \sqrt{7+9} = 4$

Ex.14 The eccentricity of the ellipse represented by the equation $25x^2 + 16y^2 - 150x - 175 = 0$ is-

(A) 2/5 (B) 3/5

(C) 4/5 (D) None of these

Sol.[B] $25(x^2 - 6x + 9) + 16y^2 = 175 + 225$

or $25(x-3)^2 + 16y^2 = 400$ or $\frac{X^2}{16} + \frac{Y^2}{25} = 1$. (b > a)

Form $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$

$\therefore$ Major axis lies along y- axis. ;

$\therefore e = \sqrt{1-\frac{a^2}{b^2}} = 1 - \sqrt{\frac{16}{25}}$;

$\therefore e = \frac{3}{5}$

Ex.15 For what value of λ does the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$.

Sol. $\because$ Equation of ellipse is

$9x^2 + 16y^2 = 144$ or $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Comparing this with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

then we get $a^2 = 16$ and $b^2 = 9$

& comparing the line $y = x + \lambda$ with $y = mx + c$

$\therefore$ m = 1 and $c = \lambda$

If the line $y = x + \lambda$ touches the ellipse

$9x^2 + 16y^2 = 144$, then

$c^2 = a^2m^2 + b^2$

$\Rightarrow \lambda^2 = 16 \times 1^2 + 9$

$\Rightarrow \lambda^2 = 25$

$\therefore \lambda = \pm 5$

Ex.16 Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line $y + 2x = 4$.

Sol. Let m be the slope of the tangent, since the tangent is perpendicular to the line $y + 2x = 4$.

$\therefore m \times -2 = -1$ $\Rightarrow m = \frac{1}{2}$

Since $3x^2 + 4y^2 = 12$

or $\frac{x^2}{4} + \frac{y^2}{3} = 1$

Comparing this with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore a^2 = 4$ and $b^2 = 3$

So the equation of the tangents are

$$y = \frac{1}{2}x \pm \sqrt{4x\frac{1}{4}+3}$$

$\Rightarrow y = \frac{1}{2}x \pm 2$ or $x - 2y \pm 4 = 0$