HYPERBOLA

(KEY CONCEPTS + SOLVED EXAMPLES)

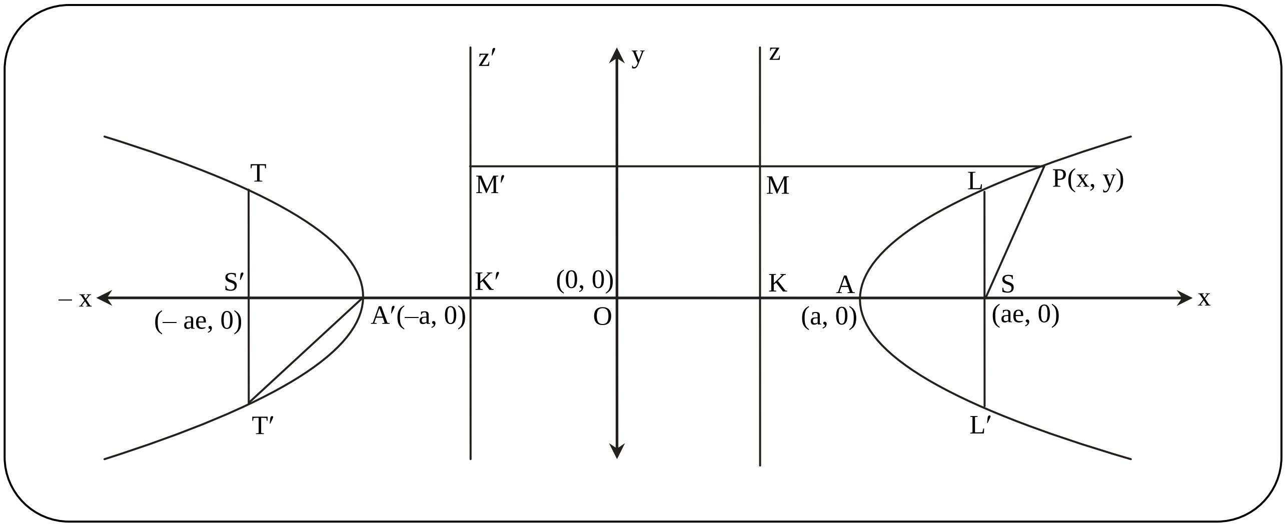
HYPERBOLA

1. *Standard equation and definitions*
2. *Conjugate Hyperbola*
3. *Parametric equation of the Hyperbola*
4. *Position of a point $P(x_1, y_1)$ with respect to Hyperbola*
5. *Line and Hyperbola*
6. *Equation of the Tangent*

KEY CONCEPTS

1. Standard Equation and Definitions

Standard Equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



(i) Definition hyperbola :

A **Hyperbola** is the locus of a point in a plane which moves in the plane in such a way that the ratio of its distance from a fixed point (called focus) in the same plane to its distance from a fixed line (called directrix) is always constant which is always greater than unity.

(ii) Vertices :

The point A and A′ where the curve meets the line joining the foci S and S′ are called vertices of hyperbola.

(iii) Transverse and Conjugate axes :

The straight line joining the vertices A and A′ is called transverse axes of the hyperbola. Straight line perpendicular to the transverse axes and passes through its centre called conjugate axes.

(iv) Latus Rectum :

The chord of the hyperbola which passes through the focus and is perpendicular to its transverse axes is called latus rectum. Length of latus rectum = $\frac{2b^2}{a}$.

(v) Eccentricity :

For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $b^2 = a^2(e^2 - 1)$

$$e = \sqrt{1+\left(\frac{2b}{2a}\right)^2} = \sqrt{1+\left(\frac{\text{Conjugate axes}}{\text{Transverse axes}}\right)^2}$$

(vi) Focal distance :

The distance of any point on the hyperbola from the focus is called the focal distance of the point.

Note : The difference of the focal distance of a point on the hyperbola is constant and is equal to the length of the transverse axes. $|S'P - SP| = 2a$ (const.)

2. Conjugate Hyperbola

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called conjugate hyperbola.

Equation of conjugate hyperbola $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Note :

(i) If e_1 and e_2 are the eccentricities of the hyperbola and its conjugate then $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$

(ii) The focus of hyperbola and its conjugate are concyclic.

S.No.	Particulars	Hyperbola	Conjugate Hyperbola
		$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
1.	Co-ordinate of the centre	(0, 0)	(0, 0)
2.	Co-ordinate of the vertices	(a, 0) & (–a, 0)	(0, b) & (0, –b)
3.	Co-ordinate of foci	$(\pm ae, 0)$	$(0, \pm be)$
4.	Length of the transverse axes	2a	2b
5.	Length of the conjugate axes	2b	2a
6.	Equation of directrix	$x = \pm a/e$	$y = \pm b/e$
7.	Eccentricity	$e=\sqrt{1+\frac{b^2}{a^2}}$	$e=\sqrt{1+\frac{a^2}{b^2}}$
8.	Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
9.	Equation of transverse axes	y = 0	x = 0
10.	Equation of conjugate axes	x = 0	y = 0

3. Parametric equation of the Hyperbola

Let the equation of ellipse in standard form will be given by $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Then the equation of ellipse in the parametric form will be given by $x = a\sec\phi$, $y = b\tan\phi$ where ϕ is the eccentric angle whose value vary from $0 \le \phi < 2\pi$. Therefore coordinate of any point P on the ellipse will be given by $(a\sec\phi, b\tan\phi)$.

4. Position of a point P(x_1, y_1) with respect to Hyperbola

The quantity $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1$ is positive, zero or negative according as the point (x_1, y_1) lies inside on or outside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

5. Line and Hyperbola

''The straight line $y = mx + c$ is a sacant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ according as $c^2 > = < a^2m^2 - b^2$

6. Equation of Tangent

(i) The equation of tangents of slope m to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $y = mx \pm \sqrt{a^2m^2-b^2}$ and the co-ordinates of the point of contacts are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}},\pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$$

(ii) Equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$

(iii) Equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(a\sec\theta, b\tan\theta)$ is $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$

Note : In general two tangents can be drawn from an external point (x_1, y_1) to the hyperbola and they are $y - y_1 = m_1(x - x_1)$ and $y - y_1 = m_2(x - x_1)$, where m_1 and m_2 are roots of

$(x_1^2 - a^2)m^2 - 2x_1y_1 + y_1^2 + b^2 = 0$

SOLVED EXAMPLES

Ex.1 Find the equation of the hyperbola whose directrix is 2x + y = 1, focus (1, 2) and eccentricity $\sqrt{3}$.

Sol. Let P (x,y) be any point on the hyperbola. Draw PM perpendicular from P on the directrix.

Then by definition

SP = e PM

$\Rightarrow (SP)^2 = e^2(PM)^2$

$\Rightarrow (x-1)^2 + (y-2)^2 = 3\left\{\frac{2x+y-1}{\sqrt{4+1}}\right\}^2$

$\Rightarrow 5(x^2 + y^2 - 2x - 4y + 5\}$

$= 3(4x^2 + y^2 + 1 + 4xy - 2y - 4x)$

$\Rightarrow 7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$

which is the required hyperbola.

Ex.2 Find the lengths of transverse axis and conjugate axis, eccentricity and the co-ordinates of foci and vertices; lengths of the latus rectum, equations of the directrices of the hyperbola $16x^2 - 9y^2 = -144$.

Sol. The equation $16x^2 - 9y^2 = -144$ can be written as

$\frac{x^2}{9} - \frac{y^2}{16} = -1$. This is of the form

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

$\therefore$ $a^2 = 9, b^2 = 16 \Rightarrow a = 3,\ b = 4$

Length of transverse axis :

The length of transverse axis = 2b = 8

Length of conjugate axis :

The length of conjugate axis = 2a = 6

Eccentricity : $e = \sqrt{1+\frac{a^2}{b^2}} = \sqrt{1+\frac{9}{16}} = \frac{5}{4}$

Foci : the co- ordinates of the foci are (0,± be), i.e., (0, ± 4)

Length of Latus rectum :

The length of latus rectum = $\frac{2a^2}{b} = \frac{2(3)^2}{4} = \frac{9}{2}$

Equation of directrices :

The equation of directrices are $y = \pm\frac{b}{e}$

$y = \pm\frac{4}{(5/4)} = \pm\frac{16}{5}$

Ex.3 Find the position of the point (5, – 4) relative to the hyperbola $9x^2 - y^2 = 1$.

Sol. Since $9(5)^2 - (-4)^2 - 1 = 225 - 16 - 1 = 208 > 0$ so the point (5, –4) lies outside the hyperbola $9x^2 - y^2 = 1$

Ex.4 The line 5x + 12y = 9 touches the hyperbola $x^2 - 9y^2 = 9$ at the point

(A) (– 5,4/3)

(B) (5, – 4/3)

(C) (3, – 1/2)

(D) None of these

Sol.[B] We have : m = Slope of the tangent = $-\frac{5}{12}$

If a line of slope m is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the coordinates of the point of contact are

$\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$

Here, $a^2 = 9$, $b^2 = 1$ and m = – 5/12

So, points of contact are $\left(\pm 5, \pm\frac{4}{3}\right)$

i.e. $\left(-5, \frac{4}{3}\right)$ and $\left(5, -\frac{4}{3}\right)$.

Out of these two points $\left(5, -\frac{4}{3}\right)$ lies on the line 5x + 12y = 9. Hence, $\left(5, -\frac{4}{3}\right)$ is the required point.

Ex. 5 The equation of the common tangents to the parabola $y^2 = 8x$ and the hyperbola $3x^2 - y^2 = 3$ is -

(A) $2x \pm y + 1 = 0$ (B) $x \pm y + 1 = 0$

(C) $x \pm 2y + 1 = 0$ (D) $x \pm y + 2 = 0$

Sol.[A] Parabola $y^2 = 8x$

$\therefore$ $4a = 8 \Rightarrow a = 2$

Any tangent to the parabola is

$$y = mx + \frac{2}{m} \qquad ...(i)$$

If it is also tangent to the hyperbola

$\frac{x^2}{1} - \frac{y^2}{3} = 1$ i.e. $a^2 = 1, b^2 = 3$ then

$$c^2 = a^2m^2 - b^2 \Rightarrow \left(\frac{2}{m}\right)^2 = 1.m^2 - 3$$

or $m^4 - 3m^2 - 4 = 0 \Rightarrow (m^2 - 4)(m^2 + 1) = 0$

$\therefore$ $m = \pm 2$ putting for m in (i), we get the tangents as $2x \pm y + 1 = 0$

Ex.6 The locus of the point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}k = 0$ and $\sqrt{3}kx + ky - 4\sqrt{3} = 0$ for different values of k is -

(A) Ellipse

(B) Parabola

(C) Circle

(D) Hyperbola

Sol.[D] $\sqrt{3}x - y = 4\sqrt{3}k$ ...(i)

and $\sqrt{3}kx + ky - 4\sqrt{3} = 0$

$\Rightarrow k(\sqrt{3}x + y) = 4\sqrt{3}$ ...(ii)

To find the locus of their point of intersection eliminate the variable K between the equations from (i) $K = \frac{\sqrt{3}x - y}{4\sqrt{3}}$ and putting in (ii), we get

$(\sqrt{3}x - y)(\sqrt{3}x + y) = (4\sqrt{3})^2$

$3x^2 - y^2 = 48$

or $\frac{x^2}{16} - \frac{y^2}{48} = 1$

Hence the locus is hyperbola

Ex.7 The eccentricity of the conic represented by $x^2 - y^2 - 4x + 4y + 16 = 0$ is -

(A) 1 (B) $\sqrt{2}$

(C) 2 (D) $\frac{1}{2}$

Sol.[B] We have $x^2 - y^2 - 4x + 4y + 16 = 0$

or $(x^2 - 4x) - (y^2 - 4y) = -16$

or $(x^2 - 4x + 4) - (y^2 - 4y + 4) = -16$

or $(x - 2)^2 - (y - 2)^2 = -16$

or $\frac{(x-2)^2}{4^2} - \frac{(y-2)^2}{4^2} = -1$

i.e. $e^2 = 1 + \frac{a^2}{b^2}$ ($\because$ conjugate hyperbola)

$$e^2 = 1 + \frac{4^2}{4^2} \Rightarrow e = \sqrt{2}$$

Ex.8 The equation $9x^2 - 16y^2 - 18x + 32y - 151 = 0$ represent a hyperbola -

(A) The length of the transverse axes is 4

(B) Length of latus rectum is 9

(C) Equation of directrix is $x = \frac{21}{5}$ and $x = -\frac{11}{5}$

(D) None of these

Sol.[C] We have $9x^2 - 16y^2 - 18x + 32y - 151 = 0$

$9(x^2 - 2x) - 16(y^2 - 2y) = 151$

$9(x^2 - 2x + 1) - 16(y^2 - 2y + 1) = 144$

$9(x - 1)^2 - 16(y - 1)^2 = 144$

$$\frac{(x-1)^2}{16} - \frac{(y-1)^2}{9} = 1$$

Comparing with $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$

where $X = x - 1$, $Y = y - 1$

and $a^2 = 16, b^2 = 9$ so

The length of the transverse axes $= 2a = 8$

The length of the latus rectum $= \frac{2b^2}{a} = \frac{9}{2}$

The equation of the directrix $X = \pm \frac{a}{e}$

$x - 1 = \pm \frac{16}{5} \Rightarrow x = \pm \frac{16}{5} + 1$

$$x = \frac{21}{5}; x = -\frac{11}{5}$$

Ex.9 For what value of λ does the line $y = 2x + \lambda$ touches the hyperbola $16x^2 - 9y^2 = 144$?

Sol. $\because$ Equation of hyperbola is $16x^2 - 9y^2 = 144$

or $\frac{x^2}{9} - \frac{y^2}{16} = 1$ comparing this with

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get $a^2 = 9$, $b^2 = 16$ and

comparing this line $y = 2x + \lambda$ with $y = mx + c$;

$m = 2$ & $c = \lambda$

If the line $y = 2x + \lambda$ touches the hyperbola

$16x^2 - 9y^2 = 144$

then $c^2 = a^2m^2 - b^2 \quad \Rightarrow \lambda = 9(2)^2 - 16$

$\therefore \lambda = \pm 2\sqrt{5}$

Ex.10 Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

Sol. Let m be the slope of the tangent since the tangent is perpendicular to the line $x - y + 4 = 0$.

$\therefore m \times 1 = -1 \qquad \Rightarrow m = -1$

since $\qquad x^2 - 4y^2 = 36$

or $\frac{x^2}{36} - \frac{y^2}{9} = 1$

Comparing this with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$;

$\therefore a^2 = 36$ & $b^2 = 9$ so the equation of tangents are

$y = (-1)\,x \pm \sqrt{36 \times (-1)^2 - 9}$

$\Rightarrow y = -x \pm \sqrt{27}$ or $x + y \pm 3\sqrt{3} = 0$