PROPERTIES

OF

TRIANGLE

(KEY CONCEPTS & SOLVED EXAMPLES)

PROPERTIES OF TRIANGLE

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KEY CONCEPTS

1. Introduction

A triangle has three sides and three angles. In this chapter we shall find the relation between the sides and trigonometrical ratios of angles of a triangle. We shall denote the angle BAC, CBA and ACB by A, B, C, and the corresponding sides opposite to them by a, b and c respectively. These six elements of a triangle are connected by the following relations

(i) $A + B + C = 180^\circ$ or π

(ii) $a + b > c$, $b + c > a$, $c + a > b$

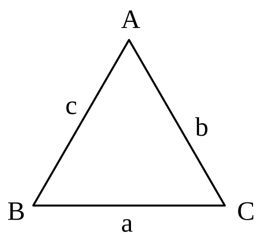
(iii) $a > 0$, $b > 0$, $c > 0$

2. Sine rule

The sides of a triangle (any type of triangle) are proportional to the sines of the angle opposite to them.

In triangle ABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Note :

(i) The above rule may also be expressed as

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

(ii) The sine rule is very useful tool to express sides of a triangle in terms of sines of angle and vice-versa in the following manner :

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \ k \text{ (Let)}$$

$\Rightarrow$ $a = k \sin A$, $b = k \sin B$, $c = k \sin C$

similarly, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} \ \lambda$ (Let)

$\Rightarrow$ $\sin A = \lambda a$, $\sin B = \lambda b$, $\sin C = \lambda c$

3. Cosine rule

In any triangle ABC

(i) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

(ii) $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

(iii) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

4. Projection formulae

In any ΔABC ;

(i) $a = b \cos C + c \cos B$

(ii) $b = c \cos A + a \cos C$

(iii) $c = a \cos B + b \cos A$

i.e. any side of a triangle is equal to the sum of the projection of other two sides on it.

5. Napier's Analogy (Tangent rule)

In any Δ ABC,

(i) $\tan\left(\dfrac{B - C}{2}\right) = \left(\dfrac{b - c}{b + c}\right) \cot\left(\dfrac{A}{2}\right)$

(ii) $\tan\left(\dfrac{A - B}{2}\right) = \left(\dfrac{a - b}{a + b}\right) \cot\left(\dfrac{C}{2}\right)$

(iii) $\tan\left(\dfrac{C - A}{2}\right) = \left(\dfrac{c - a}{c + a}\right) \cot\left(\dfrac{B}{2}\right)$

6. Trigonometrical ratios of the half angles of a triangle

If the perimeter of a triangle ABC is denoted by 2s then

$2s = a + b + c$

and area denoted by Δ . Then

6.1 Formulae for sin $\left(\frac{A}{2}\right)$, sin $\left(\frac{B}{2}\right)$, sin $\left(\frac{C}{2}\right)$

In any Δ ABC

(i) $\sin\left(\dfrac{A}{2}\right) = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$

(ii) $\sin\left(\dfrac{B}{2}\right) = \sqrt{\dfrac{(s-c)(s-a)}{ac}}$

(iii) $\sin\left(\dfrac{C}{2}\right) = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$

6.2 Formulae for $\cos\left(\frac{A}{2}\right)$, $\cos\left(\frac{B}{2}\right)$, $\cos\left(\frac{C}{2}\right)$

In any Δ ABC

(i) $\cos\left(\frac{A}{2}\right) = \sqrt{\frac{s(s-a)}{bc}}$

(ii) $\cos\left(\frac{B}{2}\right) = \sqrt{\frac{s(s-b)}{ac}}$

(iii) $\cos\left(\frac{C}{2}\right) = \sqrt{\frac{s(s-c)}{ab}}$

6.3 Formulae for $\tan\left(\frac{A}{2}\right)$, $\tan\left(\frac{B}{2}\right)$, $\tan\left(\frac{C}{2}\right)$

In any Δ ABC

(i) $\tan\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

(ii) $\tan\left(\frac{B}{2}\right) = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

(iii) $\tan\left(\frac{C}{2}\right) = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

7. Area of triangle

In a triangle ABC

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{2\Delta}{abc}$$

The area of Δ ABC is given by

(i) $\Delta = \frac{1}{2} bc \sin A$ (ii) $\Delta = \frac{1}{2} ca \sin B$

(iii) $\Delta = \frac{1}{2} ab \sin C$

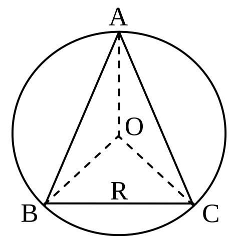
7.1 Hero's Formula :

In any ΔABC

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

8. Circumcircle of a triangle and its radius

The circle which passes through the angular points of a triangle is called its circumcircle. In a triangle the point of intersection of perpendicular bisector of the sides and is called the circumcentre. Its radius is always denoted by R.



The circumcentre may lie within, outside or upon one of the sides of the triangle. In a right angled triangle the circumcentre is the midpoint of the hypotenuse.

In a triangle ABC, circumradius is given by

(i) $R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$

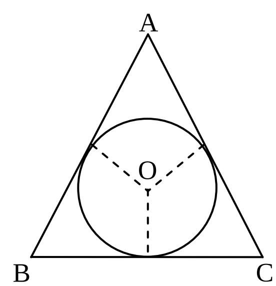
(ii) $R = \frac{abc}{4\Delta}$

(iii) $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{2R}$

9. Inscribed circle or in circle of a triangle and it's radius

Incircle or Inscribed circle :

The circle which can be inscribed with in a triangle and touch each of the sides is called its inscribed circle or incircle. The centre of this circle is the point of intersection of the bisector of the angle of the triangle.



The radius of this circle is always denoted by r and is equal to the length of the perpendicular from its centre to any one of the sides of triangle.

In - Radius : The radius r of the inscribed circle of a triangle ABC is given by

(i) $r = \frac{\Delta}{s}$

(ii) $r = (s-a)\tan\left(\frac{A}{2}\right)$, $r = (s-b)\tan\left(\frac{B}{2}\right)$

and $r = (s-c)\tan\left(\frac{C}{2}\right)$

(iii) $r = \dfrac{a\sin\left(\frac{B}{2}\right).\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{A}{2}\right)}$,

$r = \dfrac{b\sin\left(\frac{A}{2}\right).\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{B}{2}\right)}$ & $r = \dfrac{c\sin\left(\frac{B}{2}\right).\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{C}{2}\right)}$

(iv) $r = 4R\sin\left(\frac{A}{2}\right).\sin\left(\frac{B}{2}\right).\sin\left(\frac{C}{2}\right)$

10. Escribed circles of a triangle and their radi

The circle which touches the side BC and two sides AB and AC produced of a triangle ABC is called the Escribed circle opposite to the angle A. Its radius is denoted by r_1. Similarly, r_2 and r_3 denote the radii of the escribed circle opposite to the angle B and C respectively.

The centres of the escribed circle are called the Ex-centres. The centre of the escribed circles opposite to the angle A is the point of intersection of the external bisector of angle B and C. The internal bisector of angle A also passes through the same point. The centre is generally denoted by I_1.

Radii of Ex-circles : In any ΔABC,

(i) $r_1 = \dfrac{\Delta}{s-a}$, $r_2 = \dfrac{\Delta}{s-b}$, $r_3 = \dfrac{\Delta}{s-c}$

(ii) $r_1 = s\tan\dfrac{A}{2}$, $r_2 = s\tan\dfrac{B}{2}$, $r_3 = s\tan\dfrac{C}{2}$

(iii) $r_1 = \dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$,

$r_2 = \dfrac{b\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}}$,

$r_3 = \dfrac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

(iv) $r_1 = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$,

$r_2 = 4R\cos\dfrac{A}{2}\sin\dfrac{B}{2}\cos\dfrac{C}{2}$,

$r_3 = 4R\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2}$

SOLVED EXAMPLES

Ex.1 In a triangle ABC, if a = 3, b = 4 & sin A = $\frac{3}{4}$, then $\angle$ B =

(A) 60º (B) 90º (C) 45º (D) 30º

Sol. We have, $\frac{\sin A}{a} = \frac{\sin B}{b}$

or, $\sin B = \frac{b}{a} \sin A$

since, a = 3, b = 4 , sin A = $\frac{3}{4}$,

we get, $\sin B = \frac{4}{3} \times \frac{3}{4} = 1$

$\therefore$ $\angle B = 90^\circ$ **Ans.[B]**

Ex.2 If A = 75º, B = 45º, then $b + c\sqrt{2}$ =

(A) a (B) a + b + c

(C) 2a (D) $\frac{1}{2}$ (a + b + c)

Sol. C = 180º – 120º = 60º

Use sine rule

$$\frac{a}{\sin 75^\circ} = \frac{b}{\sin 45^\circ} = \frac{c}{\sin 60^\circ} = k$$

$\Rightarrow (b + c\sqrt{2}) = k(\sin 45^\circ + \sqrt{2}\sin 60^\circ)$

$$= k\,\frac{\sqrt{3}+1}{\sqrt{2}} = 2k\,\frac{\sqrt{3}+1}{2\sqrt{2}}$$

$= 2k\sin 75^\circ = 2k \sin A = 2a$

Ans.[C]

Ex.3 The smallest angle of the triangle whose sides are $6+\sqrt{12}$, $\sqrt{48}$, $\sqrt{24}$ is

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Sol. Let $a = 6+\sqrt{12}$, $b = \sqrt{48}$, $c = \sqrt{24}$

Here c is the smallest side.

$\angle$C is the smallest angle of the triangle.

Now $\cos C = \frac{a^2+b^2-c^2}{2ab}$

$$= \frac{(48+24\sqrt{3})+48-24}{4(3+\sqrt{3})4\sqrt{3}} = \frac{\sqrt{3}}{2}$$

so, $\angle C = \pi/6$ **Ans.[B]**

Ex.4 In a triangle, $a^2 + b^2 + c^2 = ca + ab\sqrt{3}$. Then the triangle is-

(A) Equilateral

(B) Right angled and isosceles

(C) Right angled with A = 90º, B = 60º, C = 30º

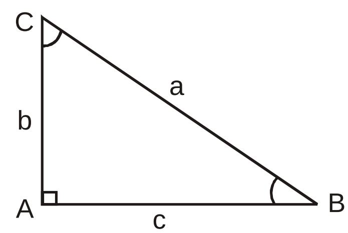
(D) None of the above

Sol. We have

$a^2 + b^2 + c^2 = ca + ab\sqrt{3}$

$\Rightarrow a^2 + b^2 + c^2 - ca - ab\sqrt{3} = 0$

$$\Rightarrow \left(\frac{a\sqrt{3}}{2} - b\right)^2 + \left(\frac{a}{2} - c\right)^2 = 0$$



It is possible only when

$\frac{a\sqrt{3}}{2} - b = 0$ & $\frac{a}{2} - c = 0$

$\Rightarrow \sqrt{3}a = 2b = 2c\sqrt{3} = k$ (let)

$\Rightarrow a = \frac{k}{\sqrt{3}}$, $b = \frac{k}{2}$, $c = \frac{k}{2\sqrt{3}}$

$\because b^2 + c^2 = a^2$

$\therefore \angle A = 90^\circ$

$\therefore \sin B = \frac{b}{a} = \frac{\sqrt{3}}{2}$

$\angle B = 60^\circ$, $\angle C = 30^\circ$ **Ans.[C]**

Ex.5 In a ΔABC, 2s = perimeter and R circum-radius. Then s/R is equal to-

(A) sin A + sin B + sin C

(B) cos A + cos B + cos C

(C) $\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}$

(D) None of these

Sol. $\frac{s}{R} = \frac{(a+b+c)}{2R} = \frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R}$

= sin A + sin B + sin C **Ans.[A]**

Ex.6 The diameter of the circum-circle of a triangle with sides 5 cm, 6 cm and 7 cm is -

(A) $\frac{3\sqrt{6}}{2}$ cm (B) $2\sqrt{6}$ cm

(C) $\frac{35}{48}$ cm (D) None of these

Sol. Radius of circum-circle is given by $R = \frac{abc}{4\Delta}$

and $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

where $s = \frac{a+b+c}{2}$

Here a = 5 cm, b = 6 cm, and c = 7 cm

$\therefore s = \frac{5+6+7}{2} = 9$

$\Delta = \sqrt{9(9-5)(9-6)(9-7)} = \sqrt{216} = 6\sqrt{6}$

$\Rightarrow R = \frac{5.6.7}{4.6\sqrt{6}} = \frac{35}{4\sqrt{6}}$

$\Rightarrow D = \frac{35}{2\sqrt{6}}$ **Ans.[D]**

Ex.7 If R denotes circum-radius then in a ΔABC, $\frac{b^2-c^2}{2aR}$ is equal to -

(A) cos (B – C) (B) sin (B – C)

(C) cos B – cos C (D) None of these

Sol. $\frac{b^2-c^2}{2aR}$

$= \frac{4R^2(\sin^2 B - \sin^2 C)}{4R^2 \sin A}$

$= \frac{\sin(B+C)\sin(B-C)}{\sin A}$

[$\because$ sin (B + C) = sin (π – A) = sin A]

= sin (B – C) **Ans.[B]**

Ex.8 In a ΔABC, the sides are in the ratio 4 : 5 : 6. The ratio of the circum-radius and the in-radius is -

(A) 8 : 7 (B) 3 : 2

(C) 7 : 3 (D) 16 : 7

Sol. Here a = 4k, b = 5k, c = 6k

$\therefore s = \frac{15k}{2}$

$\therefore \Delta = \sqrt{\frac{15k}{2}\left(\frac{15k}{2}-4k\right)\left(\frac{15k}{2}-5k\right)\left(\frac{15k}{2}-6k\right)}$

$= \frac{15\sqrt{7}}{4} k^2$

But $R = \frac{abc}{4\Delta} = \frac{4k.5k.6k}{15\sqrt{7}k^2} = \frac{8}{\sqrt{7}} k$

and $r = \frac{\Delta}{s} = \frac{15\sqrt{7}}{4} k^2 . \frac{2}{15k} = \frac{\sqrt{7}}{2} k$

$\therefore \frac{R}{r} = \frac{\frac{8k}{\sqrt{7}}}{\frac{\sqrt{7}k}{2}} = \frac{16}{7} = 16 : 7$ **Ans.[D]**

Ex.9 The ratio of the circum-radius and in-radius of an equilateral triangle is-

(A) 3 : 1 (B) 1 : 2

(C) 2 : $\sqrt{3}$ (D) 2 : 1

Sol. $\frac{r}{R} = \frac{a\cos A + b\cos B + c\cos C}{a+b+c}$

In equilateral triangle A = B = C = 60º

$= \frac{(a+b+c)\cos 60^\circ}{a+b+c} = \frac{1}{2}$ **Ans.[D]**

Ex.10 A ΔABC is right angled at B. Then the diameter of the in-circle of the triangle is-

(A) 2(c + a – b) (B) c + a – 2b

(C) c + a – b (D) None of these

Sol. $r = \frac{\Delta}{s} = \frac{\left(\frac{1}{2}\right). ac}{\left(\frac{1}{2}\right).(a+b+c)} = \frac{ac}{(a+b+c)}$

$= \frac{ac(c+a-b)}{(c+a)^2 - b^2} = \frac{ac(c+a-b)}{c^2+2ca+a^2-b^2}$

$= \frac{ac(c+a-b)}{2ca+b^2-b^2} = \frac{c+a-b}{2}$

($\because a^2 + c^2 = b^2$) **Ans.[C]**

Ex.11 In an equilateral triangle, the in-radius, circum-radius and one of the ex-radii are in the ratio-

(A) 2 : 3 : 5 (B) 1 : 2 : 3

(C) 3 : 7 : 9 (D) 3 : 7 : 9

Sol. We have, $\Delta = \frac{\sqrt{3}}{4}a^2$, $s = \frac{3a}{2}$

$\therefore \quad r = \frac{\Delta}{s} = \frac{a}{2\sqrt{3}}$,

$R = \frac{abc}{4\Delta} = \frac{a^3}{\sqrt{3}a^2} = \frac{a}{\sqrt{3}}$

and $r_1 = \frac{\Delta}{s-a} = \frac{\sqrt{3}/4a^2}{a/2} = \frac{\sqrt{3}}{2}a$

Hence, $r : R : r_1 = \frac{a}{2\sqrt{3}} : \frac{a}{\sqrt{3}} : \frac{\sqrt{3}}{2}a$

= 1: 2 : 3 **Ans.[B]**

Ex.12 If in a ΔABC, $8R^2 = a^2 + b^2 + c^2$, then the triangle ABC is-

(A) right angled (B) isosceles

(C) equilateral (D) None of these

Sol. We have, $8R^2 = a^2 + b^2 + c^2$

$\Rightarrow 8R^2 = 4R^2 (\sin^2 A + \sin^2 B + \sin^2 C)$

$\Rightarrow \sin^2 A + \sin^2 B + \sin^2 C = 2$

$\Rightarrow 1 - \cos^2 A + 1 - \cos^2 B + \sin^2 C = 2$

$\Rightarrow (\cos^2 A - \sin^2 C) + \cos^2 B = 0$

$\Rightarrow \cos (A + C) \cos (A - C) + \cos^2 B = 0$

$\Rightarrow - \cos B \{\cos (A - C) - \cos B\} = 0$

$\Rightarrow - \cos B \{\cos (A - C) + \cos (A + C) = 0$

$\Rightarrow - 2 \cos A \cos B \cos C = 0$

$\Rightarrow \cos A = 0$ or $\cos B = 0$ or $\cos C = 0$

$\Rightarrow A = \frac{\pi}{2}$ or $B = \frac{\pi}{2}$ or $C = \frac{\pi}{2}$ **Ans.[A]**

Ex.13 If the ex-radii of a triangle are in H.P. then corresponding sides are in-

(A) A.P. (B) G.P.

(C) H.P. (D) None of these

Sol. $\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}$ are in A.P.

$\Rightarrow \frac{s-a}{\Delta}, \frac{s-b}{\Delta}, \frac{s-c}{\Delta}$ are in A.P.

$\Rightarrow$ $s - a, s - b, s - c$ are in A.P.

$\Rightarrow$ $-a, -b, -c$ are in A.P. **Ans.[A]**