TRIGONOMETRICAL EQUATION

(KEY CONCEPTS + SOLVED EXAMPLES)

TRIGONOMETRICAL EQUATION

1. *Definitions*
2. *Periodic Function*
3. *General solution of standard, Trigonometrical Equations*
4. *General solutions of square of the trigonometrical equations* for your revision.

KEY CONCEPTS

1. Definition

An equation containing trigonometric function of unknown angles are known as trigonometric equations.

Ex. $\cos\theta = \frac{1}{2}$, $\tan\theta = \frac{1}{\sqrt{3}}$ and $\sin\theta = \frac{1}{2}$ etc. are trigonometric equations.

2. Periodic Function

A function f(x) is said to be periodic if there exists T > 0 such that f(x + T) = f(x) for all x in the domain of definitions of f(x). If T is the smallest positive real numbers such that f(x + T) = f(x), then it is called the period of f(x).

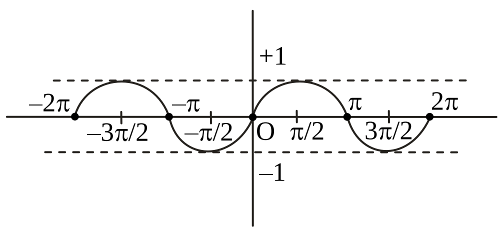
The period of sin x, cos x, sec x, cosec x is 2π and period of tan x and cot x is π.

3. General solution of standard trigonometrical equation

Since Trigonometrical functions are periodic functions, therefore, solutions of Trigonometrical equations can be generalised with the help of periodicity of Trigonometrical functions. The solution consisting of all possible solutions of a Trigonometrical equation is called its general solution.

3.1 General Solution of the equation sin θ = 0 :

By Graphical approach,



The above graph of sin θ clearly shows that $\sin\theta = 0$ at

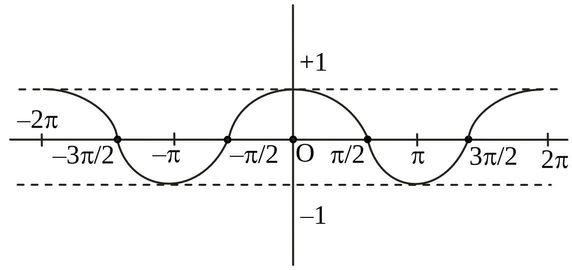
$\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi$ ....

$\sin\theta = 0$ is

$\theta = n\pi \;:\; n \in I$ i.e. n = 0, ± 1, ± 2 ....

3.2 General solution of cos θ = 0 :

By graphical approach,



The above graph of $\cos\theta$ clearly shows that $\cos\theta = 0$ at

$\theta = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2$, ...

$\cos\theta = 0$

$\theta = (2n + 1)\pi/2, \; n \in I.$

i.e. n = 0, ± 1, ± 2 ....

3.3 General solution of tan θ = 0 :

Proof: If $\tan\theta = 0$

or $\frac{\sin\theta}{\cos\theta} = 0$

$\sin\theta = 0,$

it follows that general solution of $\tan\theta = 0$ it same as of $\sin\theta = 0$

general solution of $\tan\theta = 0$ is

$\theta = n\pi; \; n \in I$

Note : General solution of $\sec\theta = 0$ and $\operatorname{cosec}\theta = 0$ does not exist because $\sec\theta$ and $\operatorname{cosec}\theta$ can never be equal to 0.

3.4 General solution of the equation sin θ = sin α :

is $\theta = n\pi + (-1)^n\alpha \;;\; n \in I$

3.5 General solution of the equation cos θ = cos α :

is $\theta = 2n\pi \pm \alpha, \; n \in I$

3.6 General solution of the equation tan θ = tan α :

is $\theta = n\pi + \alpha; \; n \in I$

4. General solution of square of the trigonometrical equations

4.1 General solution of $\sin^2\theta = \sin^2\alpha$

is $\theta = n\pi \pm \alpha \;;\; n \in I$

4.2 General solution of $\cos^2\theta = \cos^2\alpha$

is $\theta = n\pi \pm \alpha \;;\; n \in I$

4.3 General solution of $\tan^2\theta = \tan^2\alpha$:

If $\tan^2\theta = \tan^2\alpha$

$\Rightarrow \theta = n\pi \pm \alpha \;;\; n \in I$

SOLVED EXAMPLES

Ex.1 If $\cos 3x = -1$, where $0^\circ \le x \le 360^\circ$, then x =

(A) 60º, 180º, 300º (B) 180º

(C) 60º, 180º (D) 180º, 300º

Sol. If $\cos 3x = -1 = \cos (2n + 1)\pi$

or, $3x = (2n + 1)\pi$

$$x = (2n + 1)\frac{\pi}{3}$$

i.e., $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ **Ans.[A]**

Ex.2 If $\sin 3\theta = \sin\theta$, then the general value of θ is

(A) $2n\pi, (2n + 1)\frac{\pi}{3}$ (B) $n\pi, (2n + 1)\frac{\pi}{4}$

(C) $n\pi, (2n + 1)\frac{\pi}{3}$ (D) None of these

Sol. $\sin 3\theta = \sin\theta$

or, $3\theta = m\pi + (-1)^m\theta$

For (m) even i.e. $m = 2n$,

then $\theta = \frac{2n\pi}{2} = n\pi$

and for (m) odd i.e. $m = (2n + 1)$

or, $\theta = (2n +1)\frac{\pi}{4}$ **Ans.[B]**

Ex.3 The number of solutions of equation, in $5x \cos 3x = \sin 6x \cos 2x$, in the interval $[0, \pi]$ are -

(A) 3 (B) 4 (C) 5 (D) 6

Sol. The given equation can be written as

$\frac{1}{2}(\sin 8x + \sin 2x) = \frac{1}{2}(\sin 8x + \sin 4x)$

or, $\sin 2x - \sin 4x$

$\Rightarrow -2 \sin x \cos 3x = 0$

Hence $\sin x = 0$ or $\cos 3x = 0$.

That is, $x = n\pi$ ($n \in I$), or $3x = k\pi + \frac{\pi}{2}$ ($k \in I$).

Therefore, since $x \in [0, \pi]$, the given equation is satisfied if $x = 0, \pi, \frac{\pi}{6}, \frac{\pi}{2}$ or $\frac{5\pi}{6}$. **Ans.[C]**

Ex.4 The number of solutions of the equation $5 \sec\theta - 13 = 12 \tan\theta$ in $[0 , 2\pi]$ is

(A) 2 (B) 1 (C) 4 (D) 0

Sol. $5 \sec \theta - 13 = 12 \tan\theta$

or, $13 \cos \theta + 12 \sin \theta = 5$

or, $\frac{13}{\sqrt{13^2+12^2}}\cos\theta + \frac{12}{\sqrt{13^2+12^2}}\sin\theta$

$= \frac{5}{\sqrt{13^2+12^2}}$

or, $\cos(\theta - \alpha) = \frac{5}{\sqrt{313}}$,

where $\cos\alpha = \frac{13}{\sqrt{313}}$

$\therefore \theta = 2n\pi \pm \cos^{-1}\frac{5}{\sqrt{313}} + \alpha$

$= 2n\pi \pm \cos^{-1}\frac{5}{\sqrt{313}} + \cos^{-1}\frac{13}{\sqrt{313}}$

As $\cos^{-1}\frac{5}{\sqrt{313}} > \cos^{-1}\frac{13}{\sqrt{313}}$,

then $\theta \in [0, 2\pi]$, when $n = 0$ (0ne value, taking positive sign) and when $n = 1$

(One value, taking negative sign.) **Ans.[A]**

Ex.5 The general solution of

$\tan\left(\frac{\pi}{2}\sin\theta\right) = \cot\left(\frac{\pi}{2}\cos\theta\right)$ is -

(A) $\theta = 2r\pi + \frac{\pi}{2}$, $r \in Z$

(B) $\theta = 2r\pi$, $r \in Z$

(C) $\theta = 2r\pi + \frac{\pi}{2}$ and $\theta = 2r\pi$, $r \in Z$

(D) None of these

Sol. We have, $\tan\left(\frac{\pi}{2}\sin\theta\right) = \cot\left(\frac{\pi}{2}\cos\theta\right)$

$\Rightarrow \tan\left(\frac{\pi}{2}\sin\theta\right) = \tan\left(\frac{\pi}{2}-\frac{\pi}{2}\cos\theta\right)$

$\Rightarrow \frac{\pi}{2}\sin\theta = r\pi + \frac{\pi}{2} - \frac{\pi}{2}\cos\theta$, $r \in Z$

$\Rightarrow \sin\theta + \cos\theta = (2r + 1)$, $r \in Z$

$\Rightarrow \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta = \frac{2r+1}{\sqrt{2}}$, $r \in Z$

$\Rightarrow \cos\left(\theta - \frac{\pi}{4}\right) = \frac{2r+1}{\sqrt{2}}$, $r \in Z$

$\Rightarrow \cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$

$\Rightarrow \theta - \frac{\pi}{4} = 2r\pi \pm \frac{\pi}{4}$, $r \in Z$

$\Rightarrow \theta = 2r\pi \pm \frac{\pi}{4} + \frac{\pi}{4}, r \in Z$

$\Rightarrow \theta = 2r\pi, 2r\pi + \frac{\pi}{2}, r \in Z$

But $\theta = 2r\pi + \frac{\pi}{2}, r \in Z$ gives extraneous roots as it does not satisfy the given equation. Therefore $\theta = 2r\pi, r \in Z$

Ans.[B]

Ex.6 The general solution of the equation $\sec 4\theta - \sec 2\theta = 2$ is –

(A) $(2n + 1)\frac{\pi}{2}, n\pi + \frac{\pi}{10}$

(B) $(2n + 1)\frac{\pi}{2}, (2n + 1)\frac{\pi}{16}$

(C) $(2n - 1)\frac{\pi}{2}, (2n + 1)\frac{\pi}{4}$

(D) $(2n + 1)\frac{\pi}{2}, (2n + 1)\frac{\pi}{10}$

Sol. Given equation is, $\sec 4\theta - \sec 2\theta = 2$

or, $\frac{1}{\cos 4\theta} - \frac{1}{\cos 2\theta} = 2, \cos 4\theta \neq 0, \cos 2\theta \neq 0$

or, $\cos 2\theta - \cos 4\theta = 2\cos 4\theta \cos 2\theta$

or, $\cos 2\theta - \cos 4\theta = \cos 6\theta + \cos 2\theta$

or, $\cos 6\theta + \cos 4\theta = 0$

or $2\cos 5\theta \cos \theta = 0$

$\therefore$ either $\cos 5\theta = 0$ or $\cos \theta = 0$

If $\cos 5\theta = 0$, then $5\theta = (2n + 1)\pi/2$

or, $\theta = (2n + 1)\pi/10$, where $n \in I$.

& if $\cos \theta = 0$, then $\theta = (2n + 1)\pi/2$, where $n \in I$.

obviously for $\theta = (2n + 1)\pi/2$ and $\theta = (2n + 1)\pi/10$, $\cos 2\theta$ or $\cos 4\theta$ are not zero

Hence $\theta = (2n + 1)\pi/2, (2n + 1)\pi/10$ are the general solutions of the given equation. **Ans.[D]**

Ex.7 The general solution of the equation $\sin^4 x + \cos^4 x = \sin x \cos x$ is-

(A) $\left(\frac{2n+1}{4}\right)\pi; n \in I$ (B) $\left(\frac{4n+1}{4}\right)\pi; n \in I$

(C) $2n\pi + \frac{\pi}{4}; n \in I$ (D) $n\pi - \frac{\pi}{4}; n \in I$

Sol. The given equation can be written as

$4\sin^4 x + 4\cos^4 x = 4\sin x \cos x$

or, $(1 - \cos 2x)^2 + (1 + \cos 2x)^2 = 2\sin 2x$

or, $2(1 + \cos^2 2x) = 2\sin 2x$

$\Rightarrow 1 + \cos^2 2x = \sin 2x$

or, $1 + 1 - \sin^2 2x = \sin 2x$

$\Rightarrow \sin^2 2x + \sin 2x = 2$

This relation is possible if and only if $\sin 2x = 1$

or, $2x = 2n\pi + \frac{\pi}{2} \Rightarrow x = n\pi + \frac{\pi}{4}$

$= \frac{(4n+1)\pi}{4} \quad (n \in I)$ **Ans.[B]**

Ex.8 The number of solutions of the equation $|\cot x| = \cot x + \frac{1}{\sin x} \; (0 \le x \le 2\pi)$ is -

(A) 0 (B) 1 (C) 2 (D) 3

Sol. If $\cot x > 0$

then $\frac{1}{\sin x} = 0$ (impossible)

Now if $\cot x < 0$

then $-\cot x = \cot x + \frac{1}{\sin x}$

$\Rightarrow \frac{2\cos x + 1}{\sin x} = 0$

$\Rightarrow \cos x = -\frac{1}{2}$

$\Rightarrow \cos x = \cos\left(\frac{2\pi}{3}\right)$

$x = 2n\pi \pm \frac{2\pi}{3}$; $n \in I$ and $0 \le x \le 2\pi$

then $x = \frac{2\pi}{3}, \frac{4\pi}{3}$ **Ans.[C]**

Ex.9 Let n be positive integer such that $\sin\frac{\pi}{2n} + \cos\frac{\pi}{2n} = \frac{\sqrt{n}}{2}$. Then -

(A) $6 \le n \le 8$ (B) $4 < n \le 8$

(C) $6 \le n \le 8$ (D) $4 < n < 8$

Sol. $\sin\frac{\pi}{2n} + \cos\frac{\pi}{2n} = \sqrt{2}\sin\left(\frac{\pi}{2n} + \frac{\pi}{4}\right)$

or, $\sin\left(\frac{\pi}{2n} + \frac{\pi}{4}\right) = \frac{\sqrt{n}}{2\sqrt{2}}$

since $\frac{\pi}{4} < \frac{\pi}{2n} + \frac{\pi}{4} < \frac{3\pi}{4}$ for $n > 1$

or, $\frac{1}{\sqrt{2}} < \frac{\sqrt{n}}{2\sqrt{2}} \le 1$

or, $2 < \sqrt{n} \le 2\sqrt{2}$

or, $4 < n \le 8$.

If $n = 1$, L.H.S. = 1, R.H.S. = 1/2

Similarly for n = 8, $\sin\left(\frac{\pi}{16} + \frac{\pi}{4}\right) \neq 1$

$\therefore 4 < n < 8$ **Ans.[D]**