VECTOR

(KEY CONCEPTS + SOLVED EXAMPLES)

VECTOR

1. *Scalar & vector quantities*
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3. *Addition & subtraction of vectors*
4. *Relation between two parallel vectors*
5. *Collinearity of three points*
6. *Coplanar and non-coplanar vectors*
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KEY CONCEPTS

1. Scalar and Vector Quantities

A physical quantity which is completely specified by its magnitude only is called **scalar**. It is represented by a real number along with suitable unit.

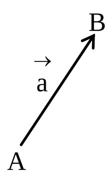
For example, Distance, Mass, Length, Time, Volume, Speed, Area are scalars.

On the other hand, a physical quantity which has magnitude as well as direction is called a **vector.** For example, Displacement, velocity, acceleration, force etc. are vector quantities.

➢ Representation of a Vector

Geometrically a vector is represented by $\vec{a}$ directed line segment. If for a vector $\vec{a}$, $\vec{a} = \overrightarrow{AB}$, then **A** is called its **initial point** and **B** is called its **terminal point**. Clearly $\overrightarrow{AB}$ and $\overrightarrow{BA}$ represents different line segments

If $\vec{a} = \overrightarrow{AB}$, then its magnitude is expressed by $|\vec{a}|$ or $|\overrightarrow{AB}|$ or AB.



Note :

The magnitude of a vector is always non negative real number.

2. Kinds of Vectors

(i) Zero or null vector :

A vector whose magnitude is zero is called **zero or null vector** and it is denoted by **0** or $\vec{0}$. The initial and terminal points of the directed line segment representing zero vector are coincident and its direction is arbitrary.

(ii) Unit vector :

A vector of unit magnitude is called a **unit vector**. A unit vector in the direction of a is denoted by $\hat{a}$. Thus

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}}{a} = \frac{\text{vector } a}{\text{magnitude of } a}$$

Note :

(i) $|\hat{a}| = 1$

(ii) Unit vectors parallel to x-axis, y-axis and z-axis are denoted by i, j and k respectively.

(iii) Two unit vectors may not be equal unless they have the same direction.

(iii) Equal Vector :

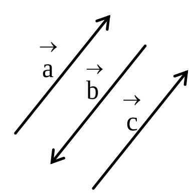
Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal , if

(a) $|\vec{a}| = |\vec{b}|$

(b) they have the same direction

(iv) Collinear vectors or Parallel vectors :

Vectors which are parallel to the same line are called collinear vectors or parallel vectors. Such vectors have either same direction or opposite direction. If they have the same direction they are said to be like vectors, and if they have opposite directions, they are called unlike vectors.



In the diagram $\vec{a}$ and $\vec{c}$ are like vectors whereas $\vec{a}$ and $\vec{b}$ are opposite vectors.

(v) Coplanar Vectors :

If the directed line segment of some given vectors lie in a plane then they are called **coplanar vectors**. It should be noted that two vectors having the same initial point are always coplanar but such three or more vectors may not be coplanar.

(vi) Position Vectors :

The vector $\overrightarrow{OA}$ which represents the position of the point A with respect to a fixed point (called origin) O is called position vector of the point A. If (x,y,z) are coordinates of the point A, then

$$\overrightarrow{OA} = x\hat{i} + y\hat{j} + z\hat{k}$$

(vii) Reciprocal vectors :

A vector which has the same direction as vector a but whose magnitude is the reciprocal of the magnitude of a, is called the reciprocal vector of vector a and is denoted by a^{-1}.

Thus if $a = \alpha\,\hat{a}$, then

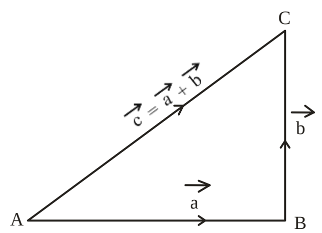
$$a^{-1} = \frac{1}{\alpha}\,.\,\hat{a} = \frac{\alpha\hat{a}}{|\vec{a}|^2} = \frac{\vec{a}}{|\vec{a}|^2}$$

Note : A unit vector is self reciprocal.

3. Addition and subtraction of Vectors

3.1 Triangle law of addition :

If two vectors are represented by two consecutive sides of a triangle then their sum is represented by the third side of the triangle but in opposite direction. This is known as the **triangle law of addition of vectors**.

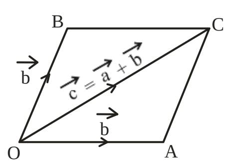


Thus, if $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{BC} = \vec{b}$, and $\overrightarrow{AC} = \vec{c}$

then $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ i.e. $\vec{a} + \vec{b} = \vec{c}$

3.2 Parallelogram Law of Addition :

If two vectors are represented by two adjacent sides of a parallelogram, then their sum is represented by the diagonal of the parallelogram whose initial point is the same as the initial point of the given vectors. This is known as **parallelogram law of addition of vectors.**



Thus if $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, and $\overrightarrow{OC} = \vec{c}$

then $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$ i.e. $\vec{a} + \vec{b} = \vec{c}$

Where $\overrightarrow{OC}$ is a diagonal of the parallelogram OABC.

3.3 Addition in component form :

If the vectors are defined in terms of $\hat{i}$, $\hat{j}$ and $\hat{k}$.

i.e. if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then their sum is defined as

$\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$

Properties of Vector Addition

Vector addition has the following properties.

(i) **Binary Operation :** The sum of two vectors is always a vector.

(ii) **Commutativity :**

For any two vectors $\vec{a}$ and $\vec{b}$,

$\vec{a} + \vec{b} = \vec{b} + \vec{c}$

(iii) **Associativity :**

For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$,

$\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$

(iv) **Identity :**

zero vector is the identity for addition For any vector a

$\vec{0} + \vec{a} = \vec{a} = \vec{a} + \vec{0}$

(v) **Additive inverse :**

For every vector its negative vector $-\vec{a}$ exists. such that

$\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$

i.e. (–a) is the additive inverse of the vector $\vec{a}$.

Also if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

then $-\vec{a} = a_1\hat{i} - a_2\hat{j} - a_3\hat{k}$

(vi) **Cancellation Law :**

For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$

$$\left.\begin{array}{l}\vec{a} + \vec{b} = \vec{a} + \vec{c} \\ \vec{b} + \vec{a} = \vec{c} + \vec{a}\end{array}\right\} \Rightarrow \vec{b} = \vec{c}$$

Subtraction of Vectors

If $\vec{a}$ and $\vec{b}$ are two vectors, then their subtraction $\vec{a} - \vec{b}$ is defined as $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

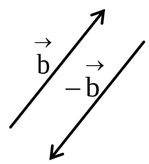
where $-\vec{b}$ is the negative of $\vec{b}$ having magnitude equal to that of $\vec{b}$ and direction opposite to $\vec{b}$.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then $\vec{a} - \vec{b} = (\vec{a}_1 - \vec{b}_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 + b_3)\hat{k}$

Note :



(i) $\vec{a} - \vec{b} \neq \vec{b} - \vec{a}$

(ii) $(\vec{a} - \vec{b}) - \vec{c} \neq \vec{a} - (\vec{b} - \vec{c})$

(iii) Since any one side of a triangle is less than the sum and greater than the difference of the other two sides, so for any two vectors $\vec{a}$ and $\vec{b}$, we have

$|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$

$|\vec{a} + \vec{b}| \geq |\vec{a}| - |\vec{b}|$

$|\vec{a} - \vec{b}| \leq |\vec{a}| + |\vec{b}|$

$|\vec{a} - \vec{b}| \geq |\vec{a}| - |\vec{b}|$

Vectors in terms of Position Vectors of end Point

If $\overrightarrow{AB}$ be any given vector and also suppose that the position vectors of initial point A and terminal point B are a and b respectively,

then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$

i.e. $\overrightarrow{AB}$ = p.v. of point B – p.v. of point A

= p.v. of terminal point – p.v. of initial point

Distance between two Points

Let A and B be two given points whose coordinate are respectively (x_1, y_1, z_1) and (x_2, y_2, z_2)

If $\vec{a}$ and $\vec{b}$ are p.v. of A and B relative to point O, then

$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

$\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

Now $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$

$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

Distance between the points and

= magnitude of $\overrightarrow{AB}$

$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Multiplication of a Vector by a Scalar

If $\vec{a}$ is a vector and m is a scalar (i.e. a real number) then ma is a vector whose magnitude is m times that of $\vec{a}$ and whose direction is the same as that of a, if m is positive and opposite to that of $\vec{a}$, if m is negative,

$\therefore$ magnitude of $m\vec{a} = |m\vec{a}|$

$\Rightarrow$ m (magnitude of a) = $m|\vec{a}|$

Again if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ then

$m\vec{a} = (ma_1)\hat{i} + (ma_2)\hat{j} + (ma_3)\hat{k}$

Note :

(i) The multiplication of a vector by a scalar is also named as 'scalar multiplication'.

(ii) From the definition of Scalar multiplication it is obvious to note that

$\vec{a} \parallel \vec{b} \Rightarrow \vec{a} = m\vec{b}$, where m is some suitable scalar.

Properties

If $\vec{a}, \vec{b}$ are any two vectors and m, n are any scalar then

(i) $m(\vec{a}) = (\vec{a})m = m\vec{a}$ (commutativity)

(ii) $m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$ (Associativity)

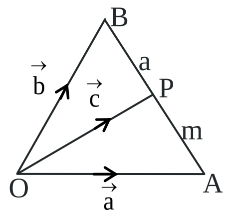
(iii) $(m + n)\vec{a} = m\vec{a} + n\vec{a}$ (Distributivity)

(iv) $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$ (Distributivity)

Position Vector of a Dividing Point

If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A and B, then the position vector c of a point P dividing AB in the ratio m : n is given by

$$\vec{c} = \frac{m\vec{b}+n\vec{a}}{m+n}$$



Particular Case :

(i) Position vector of the mid point of AB is $\frac{\vec{b}+\vec{b}}{2}$

(ii) Any vector along the internal bisector of $\angle AOB$ is given by

$\lambda(\vec{a}+\vec{b})$

Note :

(i) If the point P divides AB in the ratio m: n externally, then m/n will be negative. If m is positive and n is negative, then p.v.c of P is given by $\vec{c} = \frac{m\vec{b}-n\vec{a}}{m-n}$

(ii) If $\vec{a}, \vec{b}, \vec{c}$ are position vectors of vertices of a triangle, then p.v. of its centroid is $\frac{\vec{a}+\vec{b}+\vec{c}}{3}$

(iii) If a,b,c,d are position vectors of vertices of a tetrahedron, then p.v. of its centroid is $\frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$

4. Relation between two parallel vectors

(i) If $\vec{a}$ and $\vec{b}$ be two parallel vectors, then there exists a scalar k such that $\vec{a} = k\vec{b}$

i.e. there exist two non- zero scalar quantities x and y so that $x\vec{a} + y\vec{b} = 0$

If a and b be two non-zero non-parallel vectors then $x\vec{a} + y\vec{b} = 0 \Rightarrow x = 0$ and $y = 0$ obviously

$$x\vec{a} + y\vec{b} = 0 \Rightarrow \begin{cases} \vec{a} = 0, \vec{b} = 0 \\ \text{or} \\ x = 0, y = 0 \\ \text{or} \\ \vec{a} \parallel \vec{b} \end{cases}$$

(ii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then from the property of parallel vector, we have $\vec{a} \parallel \vec{b}$

$$\Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

5. Collinearity of three Points

(i) If $\vec{a}, \vec{b}, \vec{c}$ be position vectors of three points A,B and C respectively and x, y,z be three scalars so that all are not zero, then the necessary and sufficient conditions for three points to be collinear is that

$x\vec{a} + y\vec{b} + z\vec{c} = 0$ and $x + y + z = 0$

(ii) Three points A, B and C are collinear, if any two vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ are parallel i.e. one of them is scalar multiple of any one of the remaining vectors.

6. Coplanar & Non-Coplanar Vector

(i) If $\vec{a}, \vec{b}, \vec{c}$ be three coplanar vectors, then a vector c can be expressed uniquely as linear combination of remaining two vectors i.e.

$\vec{c} = \lambda\vec{a} + \mu\vec{b}$

Where λ and μ are suitable scalars.

Again $\vec{c} = \lambda\vec{a} + \mu\vec{b} \Rightarrow$ vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

If $\vec{a}, \vec{b}, \vec{c}$ be three coplanar vectors, then there exist three non zero scalars x, y, z so that

$x\vec{a} + y\vec{b} + z\vec{c} = 0$

(ii) If $\vec{a}, \vec{b}, \vec{c}$ be three non coplanar non zero vector then

$x\vec{a} + y\vec{b} + z\vec{c} = 0 \Rightarrow x = 0, y = 0, z = 0$

(iii) Any vector r can be expressed uniquely as the linear combination of three non coplanar and non-zero vectors $\vec{a}, \vec{b}$ and $\vec{c}$ i.e. $r = x\vec{a} + y\vec{b} + z\vec{c}$ where x, y and z are scalars.

Product of Vectors

Product of two vectors is done by two methods when the product of two vectors results in a scalar quantity then it is called **scalar product**. It is also called as **dot product** because this product is represented by putting a dot.

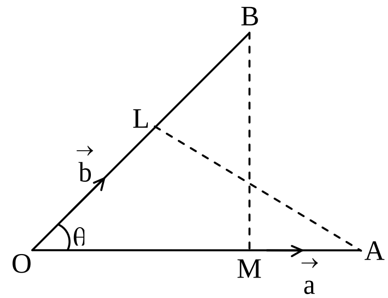
When the product of two vectors results in a vector quantity then this product is called **Vector Product**. This product is represented by (x) sign so that it is also called as **cross product.**

7. Scalar or dot product of two Vectors

7.1 Definition :

If $\vec{a}$ and $\vec{b}$ are two non zero vectors and θ be the angle between them, then their **scalar product** (or dot product) is defined as the number a b cos θ where a and b are modulii of $\vec{a}$ and $\vec{b}$ respectively and $0 \le \theta \le \pi$. It is denoted by $\vec{a} \cdot \vec{b}$. Thus

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta = a\,b\,\cos\theta$$



Note :

(i) $\vec{a} \cdot \vec{b} \in R$

(ii) $\vec{a} \cdot \vec{b} \le |\vec{a}|\,|\vec{b}|$

(iii) $\vec{a} \cdot \vec{b} > 0 \Rightarrow$ angle between $\vec{a}$ and $\vec{b}$ is acute

$\vec{a} \cdot \vec{b} < 0 \Rightarrow$ angle between $\vec{a}$ and $\vec{b}$ is obtuse.

(iv) The dot product of $\vec{a}$ zero and non- zero vector is a scalar zero.

7.2 Geometrical Interpretation :

Geometrically, the scalar product of two vectors is equal to the product of the magnitude of one and the projection of second in the direction of first vector i.e. $\vec{a} \cdot \vec{b} = \vec{a}\,(\vec{b}\cos\theta)$

$= |\vec{a}|$ (projection of $\vec{b}$ in the direction of $\vec{a}$)

Similarly $\vec{a} \cdot \vec{b} = \vec{b}\,(\vec{a}\cos\theta)$

$= |\vec{b}|$ (projection of $\vec{a}$ in the direction of $\vec{b}$)

Here projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$

Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

Scalar product in particular cases

(i) If $\vec{a}$ and $\vec{b}$ are like vectors, then θ = 0 so

$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| = \vec{a}\,\vec{b}$ i.e. scalar product of two like vectors is equal to the product of their modulii

(ii) If $\vec{a}$ and $\vec{b}$ are unlike vectors then θ = π so

$\vec{a} \cdot \vec{b} = \vec{a}\,\vec{b}\cos\pi = -\,\vec{a}\,\vec{b}$.

(iii) The scalar product of a vector by itself is equal to the square of its modulus i.e.

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2$$

(iv) If $\vec{a}$ and $\vec{b}$ are perpendicular to each other then θ = π/2, so

$\vec{a} \cdot \vec{b} = \vec{a}\,\vec{b}\cos\pi/2 = 0$

i.e. the scalar product of two perpendicular vectors is always zero.

But its converse may not be true i.e.

$\vec{a} \cdot \vec{b} = 0 \not\Rightarrow \vec{a} \perp \vec{b}$

But if a and b are non zero vectors, then

$\vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$

Thus $\vec{a} \ne 0,\ \vec{b} \ne 0,\ \vec{a} \cdot \vec{b} = 0, \Rightarrow \vec{a} \perp \vec{b}$

(v) With the help of the above cases, we get the following important results:

(a) $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ (b) $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} \cdot \hat{k} \cdot \hat{j} = 0$

(vi) If $\vec{a}$ and $\vec{b}$ are unit vectors, then

$\vec{a} \cdot \vec{b} = \cos\theta$

Properties of Scalar Product

If $\vec{a}, \vec{b}, \vec{c}$ are any vectors and m, n any scalars then

(i) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (Commutativity)

(ii) $(m\vec{a}).\vec{b} = \vec{a}.(m\vec{b}) = m(\vec{a}.\vec{b})$

(iii) $(m\vec{a}).(n\vec{b}) = (mn)(\vec{a}.\vec{b})$

(iv) $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ (Distributivity)

(v) $\vec{a}.\vec{b} = \vec{a}.\vec{c} \not\Rightarrow \vec{b} = \vec{c}$

Infact $\vec{a}.\vec{b} = \vec{a}.\vec{c} \Rightarrow \vec{a}.(\vec{b}-\vec{c}) = 0$

$\Rightarrow \vec{a} = 0$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b} - \vec{c})$

(vi) $(\vec{a}.\vec{b}).\vec{c}$ is meaningless

(vi) scalar product is not binary operation.

Note :

(a) $(\vec{a}.\vec{b}).\vec{b}$ is not defined

(b) $(\vec{a}+\vec{b})^2 = |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2$

(c) $(\vec{a}-\vec{b})^2 = |\vec{a}|^2 - 2\vec{a}.\vec{b} + |\vec{b}|^2$

(d) $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2 = |\vec{a}|^2 - |\vec{b}|^2$

(e) $|\vec{a}+\vec{b}| = |\vec{a}| + |\vec{b}| \Rightarrow \vec{a} \parallel \vec{b}$

(f) $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 \Rightarrow \vec{a} \perp \vec{b}$

(g) $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}| \Rightarrow \vec{a} \perp \vec{b}$

Scalar product in terms of Components

Let $\vec{a}$ and $\vec{b}$ be two vectors such that

$\vec{a} = \vec{a}_1\hat{i} + \vec{a}_2\hat{j} + \vec{a}_3\hat{k}$

and $\vec{b} = \vec{b}_1\hat{i} + \vec{b}_2\hat{j} + \vec{b}_3\hat{k}$

Then $\vec{a} \cdot \vec{b} = \vec{a}_1\vec{b}_1 + \vec{a}_2\vec{b}_2 + \vec{a}_3\vec{b}_3$

In particular

$\vec{a}.\vec{a} = |\vec{a}|^2 = \vec{a}_1^{\,2} + \vec{a}_2^{\,2} + \vec{a}_3^{\,2}$

For any vector $\vec{a}$,

$\vec{a} = (\vec{a}.\hat{i})\hat{i} + (\vec{a}.\hat{j})\hat{j} + (\vec{a}.\hat{k})\hat{k}$

Angle between two Vectors

(i) If $\vec{a}$ and $\vec{b}$ be two vectors and θ be the angle between them, then

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\vec{a}}{\vec{a}}.\frac{\vec{b}}{\vec{b}} = \hat{a}.\hat{b}$$

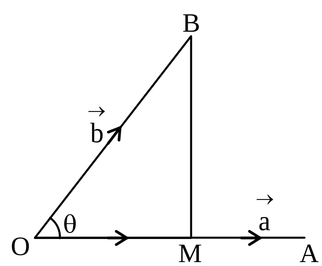
(ii) If $\vec{a} = \vec{a}_1\hat{i} + \vec{a}_2\hat{j} + \vec{a}_3\hat{k}$ and

$\vec{b} = \vec{b}_1\hat{i} + \vec{b}_2\hat{j} + \vec{b}_3\hat{k}$ then

$$\cos\theta = \frac{\vec{a}_1\vec{b}_1 + \vec{a}_2\vec{b}_2 + \vec{a}_3\vec{b}_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$$

Note : If $\vec{a}$ and $\vec{b}$ are perpendicular to each other then $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$

Components of b Along & Perpendicular to a



(i) Component along a = $\overrightarrow{OM}$

$= OM\,\hat{a} = (b\cos\theta)\,\hat{a}$

$= \dfrac{(ab\cos\theta)}{a}\hat{a}$

$= \dfrac{(a.b)}{a^2}.a$

(ii) Component perpendicular to $\vec{a} = \overrightarrow{MB}$

$= \overrightarrow{MO} + \overrightarrow{OB}$

$= \overrightarrow{OB} - \overrightarrow{OM}$

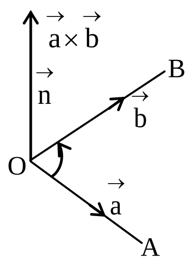
$= \vec{b} - \dfrac{(\vec{a}.\vec{b})}{a^2}.\vec{a}$

Work done by the Force

If a constant force F acting on a particle displaces it from point A to B, then work done by the force $W = F. \vec{d}$

(where $\vec{d} = \overrightarrow{AB}$)

8. Vector or cross product of two Vectors



8.1 Definition :

If $\vec{a}$ and $\vec{b}$ be two vectors and $\theta\ (0 \le \theta \le \pi)$ be the angle between them, then their vector (or cross) product is defined to be a vector whose magnitude is ab sin θ and whose direction is perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\vec{a} \times \vec{b}$ form a right handed system.

$$\therefore\ \vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$$

$$= \vec{a}\ \vec{b}\ \sin\theta\, \hat{n}$$

Where $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system.

8.2 Vector product in terms of components :

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then $\vec{a} \times \vec{b} = (a_2 b_3 - a_3b_2)\ \hat{i} + (a_3b_1 - a_1b_3)\,\hat{j}$

$+ (a_1 b_2 - a_2 b_1)\ \hat{k}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

8.3 Angle between two vectors :

If θ is the angle between $\vec{a}$ and $\vec{b}$, then

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$$

If $\hat{n}$ is the unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$, then

$$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

➤ Vector product in Particular Cases

(i) The vector product of two parallel vectors is always zero i.e. if vectors a and b are parallel, then $\vec{a} \times \vec{b} = 0$

In particular $\vec{a} \times \vec{b} = 0$

(ii) If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\ \hat{n} = \vec{a}\ \vec{b}\ \hat{n}$

(iii) If $\hat{i}, \hat{j}, \hat{k}$ be three mutually perpendicular unit vectors, then

(a) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

(b) $\hat{i} \times \hat{j} = \hat{k}$, $\hat{i} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{i}$

(c) $\hat{j} \times \hat{i} = -\hat{k}$, $\hat{k} \times \hat{j} = -\hat{i}$, $\hat{i} \times \hat{k} = -\hat{j}$

➤ Expression for sin θ

If $\vec{a} = \vec{a}_1\ \hat{i} + \vec{a}_2\ \hat{j} + \vec{a}_3\ \hat{k}$

and $\vec{b} = \vec{b}_1\ \hat{i} + \vec{b}_2\ \hat{j} + \vec{b}_3\ \hat{k}$

and θ be angle between $\vec{a}$ and $\vec{b}$, then

$$\sin^2\theta = \frac{(a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}$$

➤ Properties of Vector Product

If $\vec{a}, \vec{b}, \vec{c}$ are any vectors and m,n any scalars then

(i) $\vec{a} \times \vec{b} \ne \vec{b} \times \vec{a}$ (Non- commutativity)

but $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

and $|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{a}|$

(ii) $(m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b}) = m(\vec{a} \times \vec{b})$

(iii) $(m\vec{a}) \times (n\vec{b}) = (mn)(\vec{a} \times \vec{b})$

(iv) $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

(v) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

(Distributivity)

(vi) $\vec{a} \times \vec{b} = \vec{a} \times \vec{c} \not\Rightarrow \vec{b} = \vec{c}$ Infact

$\vec{a} \times \vec{b} = \vec{a} \times \vec{c} \Rightarrow \vec{a} \times (\vec{b} - \vec{c}) = 0$

$\Rightarrow \vec{a} = 0$ or $\vec{b} = \vec{c}$ or $\vec{a} \parallel (\vec{b} - \vec{c})$

➢ Geometrical interpretation of Vector Product

The vector product of the vectors $\vec{a}$ and $\vec{b}$ represents a vector whose modulus is equal to the area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ and $\vec{b}$. Therefore $|\vec{a} \times \vec{b}|$ = area of a parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$.

Further it should be noted that if $\vec{a}$, $\vec{b}$ represent two diagonals of a parallelogram,

then the area of the parallelogram = $\frac{1}{2}|\vec{a} \times \vec{b}|$

Note : Area of a quadrilateral ABCD

$= \frac{1}{2}|\overrightarrow{AC} \times \overrightarrow{BD}|$

➢ Area of a Triangle

(i) Area of triangle ABC = $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

(ii) If $\vec{a}, \vec{b}, \vec{c}$ are position vectors of vertices of a ΔABC then its

Area = $\frac{1}{2}|(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a})|$

Note :

Three points with position vectors $\vec{a}, \vec{b}, \vec{c}$ are collinear if

$(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a}) = 0$

➢ Moment of a Force

The moment of the force F acting at a point A about O is given by

Moment of F = $\overrightarrow{OA} \times F = \vec{r} \times F$

9. Scalar Triple Product

9.1 Definition :

If $\vec{a}, \vec{b}, \vec{c}$ are three vectors, then their scalar triple product is defined as the dot product of two vectors $\vec{a}$ and $\vec{b} \times \vec{c}$. It is generally denoted by $\vec{a} \cdot (\vec{b} \times \vec{c})$ or $[\vec{a}\ \vec{b}\ \vec{c}]$. It is read as box product of $\vec{a}, \vec{b}, \vec{c}$. Similarly other scalar triple products can be defined as $(\vec{b} \times \vec{c}).\vec{a}$, $(\vec{c} \times \vec{a}).\vec{b}$.

Note :

Scalar triple product always results in a scalar quantity (number).

9.2 Geometrical Interpretation :

The scalar triple product of three vectors is equal to the volume of the parallelopiped whose three coterminous edges are represented by the given vector.

Therefore $(\vec{a} \times \vec{b}) \cdot \vec{c} = [\vec{a}\ \vec{b}\ \vec{c}]$ = Volume of the parallelopiped whose coterminous edges are $\vec{a}$, $\vec{b}$ and $\vec{c}$

9.3 Formula for scalar Triple Product :

(i) If $\vec{a} = a_1\ell + a_2m + a_3n$, $\vec{b} = b_1\ell + b_2m + b_3n$ and $\vec{c} = c_1\ell + c_2m + c_3n$, then

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = [\ell mn]$$

(ii) $\vec{a} = \vec{a}_1\,\hat{i} + \vec{a}_2\,\hat{j} + \vec{a}_3\,\hat{k}$, $\vec{b} = \vec{b}_1\,\hat{i} + \vec{b}_2\,\hat{j} + \vec{b}_3\,\hat{k}$ and $\vec{c} = \vec{c}_1\,\hat{i} + \vec{c}_2\,\hat{j} + \vec{c}_3\,\hat{k}$, then

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(iii) For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$

(a) $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$

(b) $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}] = 0$

(c) $[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]^2$

9.4 Properties of Scalar Triple product

(i) The position of (.) and (×) can be interchanged

i.e. $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$

but $(\vec{a}\times\vec{b})\cdot\vec{c} = \vec{c}\cdot(\vec{a}\times\vec{b})$

So $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

Therefore if we don't change the cyclic order of a, b and c then the value of scalar triple product is not changed by interchanging dot and cross.

(ii) If the cyclic order of vectors is changed, then sign of scalar triple product is changed i.e.

$\vec{a}\cdot[\vec{b}\times\vec{c}] = -\vec{a}\cdot(\vec{c}\times\vec{b})$

or $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$

from (i) and (ii) we have

$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

$= -[\vec{a}\ \vec{c}\ \vec{b}] = -[\vec{b}\ \vec{a}\ \vec{c}] = -[\vec{c}\ \vec{b}\ \vec{a}]$

(iii) The scalar triple product of three vectors when two of them are equal or parallel, is zero i.e.

$[\vec{a}\ \vec{b}\ \vec{b}] = [\vec{a}\ \vec{b}\ \vec{a}] = 0$

(iv) The scalar triple product of three mutually perpendicular unit vectors is ±1 Thus

$[\hat{i}\ \hat{j}\ \hat{k}] = 1$, $[\hat{i}\ \hat{k}\ \hat{j}] = -1$

(v) If two of the three vectors $\vec{a}, \vec{b}, \vec{c}$ are parallel then $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

(vi) $\vec{a}, \vec{b}, \vec{c}$ are three coplanar vectors if $[\vec{a}\ \vec{b}\ \vec{c}] = 0$ i.e. the necessary and sufficient condition for three non-zero collinear vectors to be coplanar is

$[\vec{a}\ \vec{b}\ \vec{c}] = 0$

(vii) For any vectors $\vec{a}, \vec{b}, \vec{c}$, d

$[\vec{a}+\vec{b}\ \ \vec{c}\ \ d] = [\vec{a}\ \vec{c}\ d] + [\vec{b}\ \vec{c}\ d]$

10. Volume of Tetrahedron

(i) If $\vec{a}, \vec{b}, \vec{c}$ are position vectors of vertices A, B and C with respect to O, then volume of tetrahedron OABC

$= \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$

(ii) If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are position vectors of vertices A,B,C,D of a tetrahedron ABCD, then

$$\text{its volume} = \begin{cases} \frac{1}{6}\left[\overrightarrow{AB}\ \ \overrightarrow{AC}\ \ \overrightarrow{AD}\right] \\ \quad\text{or} \\ \frac{1}{6}\left[b-a\ \ \ c-a\ \ \ d-a\right] \end{cases}$$

11. Vector Triple Product

11.1 Definition :

The vector triple product of three vectors $\vec{a}, \vec{b}, \vec{c}$ is defined as the vector product of two vectors $\vec{a}$ and $\vec{b}\times\vec{c}$. It is denoted by $\vec{a}\times(\vec{b}\times\vec{c})$.

11.2 Properties :

(i) Expansion formula for vector triple product is given by

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

$(\vec{b}\times\vec{c})\times\vec{a} = (\vec{b}.\vec{a})\vec{c} - (\vec{c}.\vec{a})\vec{b}$.

(ii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then

$\vec{a}\times(\vec{b}\times\vec{c})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix}$$

Note :

(i) Vector triple product is a vector quantity.

(ii) $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

(iii) Vector $\vec{a} \times (\vec{b} \times \vec{c})$ is coplanar with $\vec{b}$ and $\vec{c}$

(iv) The direction of $\vec{a} \times (\vec{b} \times \vec{c})$ is perpendicular to $\vec{a}$ and parallel to and $\vec{b}$ and $\vec{c}$.

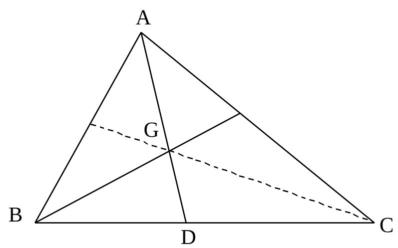
SOLVED EXAMPLE

Ex.1 If G is the centroid of triangle ABC then value of $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}$ will be-

(A) $\vec{0}$ (B) $3\overrightarrow{GA}$

(C) $3\overrightarrow{GB}$ (D) $3\overrightarrow{GC}$

Sol. If D is middle point of side BC then-



$\overrightarrow{GD}=\frac{1}{2}(\overrightarrow{GB}+\overrightarrow{GC})$

$\because$ G divides AD in the ratio of 2 : 1

$\therefore \overrightarrow{AG}=2\overrightarrow{GD}$

$\Rightarrow -\overrightarrow{GA}=\overrightarrow{GB}+\overrightarrow{GC}$

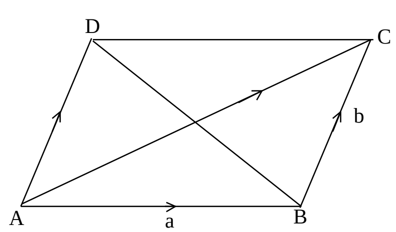
$\Rightarrow \overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=\vec{0}$ **Ans. [A]**

Ex.2 In a parallelogram ABCD, $\overrightarrow{AB}=\vec{a}$, $\overrightarrow{BC}=\vec{b}$ then $\overrightarrow{BD}$ equals-

(A) $(\vec{a}+\vec{b})$ (B) $(\vec{a}-\vec{b})$

(C) $(\vec{a}+2\vec{b})$ (D) $(\vec{b}-\vec{a})$

Sol. According to figure



$\overrightarrow{BD}=\overrightarrow{BC}+\overrightarrow{CD}$

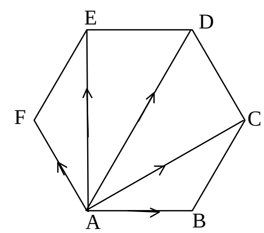
$=\vec{b}+\overrightarrow{BA}$ $[\because \overrightarrow{CD}=\overrightarrow{BA}]$

$=\vec{b}-\vec{a}$ $[\because \overrightarrow{BA}=-\vec{a}]$ **Ans.[D]**

Ex.3 If ABCDEF is a regular hexagon and $\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD}+\overrightarrow{AE}+\overrightarrow{AF}=k\,\overrightarrow{AD}$, then k equals-

(A) 2 (B) 3 (C) 6 (D) 5

Sol. $\because \overrightarrow{AB}=\overrightarrow{ED}$ and $\overrightarrow{AF}=\overrightarrow{CD}$,



so $\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD}+\overrightarrow{AE}+\overrightarrow{AF}$

$=\overrightarrow{ED}+\overrightarrow{AC}+\overrightarrow{AD}+\overrightarrow{AE}+\overrightarrow{CD}$

$=(\overrightarrow{AC}+\overrightarrow{CD})+(\overrightarrow{AE}+\overrightarrow{ED})+\overrightarrow{AD}$

$=\overrightarrow{AD}+\overrightarrow{AD}+\overrightarrow{AD}=3\overrightarrow{AD}$

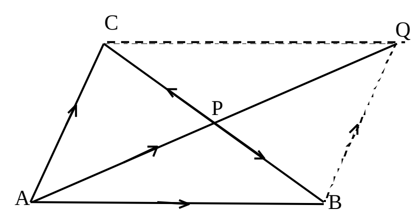
$\therefore k=3$ **Ans.[B]**

Ex.4 If a point P on the side BC of triangle ABC is such that $\overrightarrow{AP}+\overrightarrow{PB}=\overrightarrow{CP}+\overrightarrow{PQ}$ then ABQC will be-

(A) Square (B) Rectangle

(C) Parallelogram (D) None of these

Sol. From figure



$\overrightarrow{AP}+\overrightarrow{PB}=\overrightarrow{AB}$

$\overrightarrow{CP}+\overrightarrow{PQ}=\overrightarrow{CQ}$

So $\overrightarrow{AB}=\overrightarrow{CQ}$ then ABQC is a parallelogram.

Ans.[B]

Ex.5 The length of diagonal AC of a parallellogram ABCD whose two adjacent sides AB and AD are represented respectively by $2\hat{i}+4\hat{j}-5\hat{k}$ and $\hat{i}+2\hat{j}+3\hat{k}$ is-

(A) 3 (B) 4 (C) 5 (D) 7

Sol. $\because \overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{AD}$

$=3\hat{i}+6\hat{j}-2\hat{k}$

$\therefore$ Length of the diagonal $\overrightarrow{AC}=|\overrightarrow{AC}|$

$= \sqrt{3^2+6^2+(-2)^2} = 7$ **Ans.[D]**

Ex.6 If the middle points of sides BC, CA & AB of triangle ABC are respectively D, E, F then position vector of centre of triangle DEF, when position vector of A, B, C are respectively

$\hat{i}+\hat{j}$, $\hat{j}+\hat{k}$, $\hat{k}+\hat{i}$ is-

(A) $\frac{1}{3}(\hat{i}+\hat{j}+\hat{k})$ (B) $(\hat{i}+\hat{j}+\hat{k})$

(C) $2(\hat{i}+\hat{j}+\hat{k})$ (D) $\frac{2}{3}(\hat{i}+\hat{j}+\hat{k})$

Sol. The position vector of points D, E, F are respectively

$\frac{\hat{i}+\hat{j}}{2}+\hat{k}$, $\hat{i}+\frac{\hat{k}+\hat{j}}{2}$ and $\frac{\hat{i}+\hat{k}}{2}+\hat{j}$

So, position vector of centre of ΔDEF

$= \frac{1}{3}\left[\frac{\hat{i}+\hat{j}}{2}+\hat{k}+\hat{i}+\frac{\hat{k}+\hat{j}}{2}+\frac{\hat{i}+\hat{k}}{2}+\hat{j}\right]$

$= \frac{2}{3}[\hat{i}+\hat{j}+\hat{k}]$ **Ans.[D]**

Ex.7 The p.v. of the point A is $6\vec{b}-2\vec{a}$ and the point P divides any line AB in the ratio 1: 2. If the p.v. of P is $\vec{a}-\vec{b}$, then p.v. of B will be-

(A) $7\vec{a}+15\vec{b}$ (B) $7\vec{a}-15\vec{b}$

(C) $15\vec{a}-7\vec{b}$ (D) $15\vec{a}+7\vec{b}$

Sol. Let the p.v. of B be r

$\therefore \vec{a}-\vec{b} = \frac{\vec{r}+2(6\vec{b}-2\vec{a})}{1+2}$

$\Rightarrow 3\vec{a}-3\vec{b} = \vec{r}+12\vec{b}-4\vec{a}$

$\Rightarrow \vec{r} = 7\vec{a}-15\vec{b}$ **Ans.[B]**

Ex.8 If $3\hat{i}-2\hat{j}+5\hat{k}$ and $-2\hat{i}+p\hat{j}-q\hat{k}$ are collinear vectors, then-

(A) p = 4/3, q = – 10/3

(B) p = 10/3, q = 4/3

(C) p = – 4/3, q = 10/3

(D) p = 4/3, q = 10/3

Sol. Given vectors are parallel, so we have

$\frac{3}{-2} = \frac{-2}{p} = \frac{5}{-q}$

$\Rightarrow$ p = 4/3, q = 10/3 **Ans.[D]**

Ex.9 If $\overrightarrow{AO}+\overrightarrow{OB} = \overrightarrow{BO}+\overrightarrow{OC}$, then A, B, C are-

(A) coplanar (B) collinear

(C) non-collinear (D) None of these

Sol. $\overrightarrow{AO}+\overrightarrow{OB} = \overrightarrow{BO}+\overrightarrow{OC} \Rightarrow \overrightarrow{AB} = \overrightarrow{BC}$

$\therefore \overrightarrow{AB} \| \overrightarrow{BC}$ and a point B is common to both vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$. Hence A, B, C are collinear. **Ans.[B]**

Ex.10 Let A = (x + 4y) $\vec{a}$ + (2x + y + 1) $\vec{b}$ and B = (y – 2x + 2) $\vec{a}$ + (2x – 3y – 1) $\vec{b}$ where $\vec{a}$ and $\vec{b}$ are non collinear vectors, if 3A = 2B; then

(A) x = 1, y = 2 (B) x = 2, y = 1

(C) x = 2, y = – 1 (D) x = – 1, y = 2

Sol. 3 A = 3(x + 4y) $\vec{a}$ + 3 (2x + y + 1) $\vec{b}$

2 B = 2(y – 2x + 2) $\vec{a}$ + 2 (2x – 3y – 1) $\vec{b}$

$\therefore$ 3A = 2B $\Rightarrow$ 3(x + 4y) = 2 (y – 2x + 2),

3 (2x + y + 1) = 2(2x – 3y – 1)

$\Rightarrow$ 7x + 10 y = 4 and 2x + 9y = – 5

$\Rightarrow$ x = 2, y = – 1 **Ans.[C]**

Ex.11 Let position vectors of points A, B, C and D are respectively $3\hat{i}-2\hat{j}-\hat{k}$, $2\hat{i}+3\hat{j}-4\hat{k}$, $-\hat{i}+\hat{j}+2\hat{k}$ and $4\hat{i}+5\hat{j}+\lambda\hat{k}$. If the points are coplanar, then the value of λ is-

(A) $-\frac{146}{17}$ (B) $\frac{146}{17}$

(C) 0 (D) None of these

Sol. $\overrightarrow{AB} = -\hat{i}+5\hat{j}-3\hat{k}$

$\overrightarrow{AC} = -4\hat{i}+3\hat{j}+3\hat{k}$

& $\overrightarrow{AD} = \hat{i}+7\hat{j}+(\lambda+1)\hat{k}$

If A, B, C, D are coplanar, then vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar, then

$-\hat{i}+5\hat{j}-3\hat{k} = x(-4\hat{i}+3\hat{j}+3\hat{k})$

$+ y[\hat{i}+7\hat{j}+(\lambda+1)\hat{k}]$

$\Rightarrow$ – 4x + y = – 1, 3x + 7y = 5

and 3x + (λ + 1)y = – 3

Solving first two equations

$$x = \frac{12}{31},\ y = \frac{17}{31}$$

Substituting these values of x and y in third equation, we get

$$= -\frac{146}{17}$$ **Ans.[A]**

Ex.12 If $\vec{a} = \hat{i}+2\hat{j}-3\hat{k}$, $\vec{b} = \hat{i}-3\hat{j}+2\hat{k}$ and $\vec{c} = 2\hat{i}-\hat{j}+5\hat{k}$, then vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are -

(A) linearly independent

(B) collinear

(C) linearly dependent

(D) None of these

Sol. Let $x(\vec{a}) + y(\vec{b}) + z(\vec{c}) = \vec{0}$

$\Rightarrow x(\hat{i}+2\hat{j}-3\hat{k}) + y(\hat{i}-3\hat{j}+2\hat{k}) + z(2\hat{i}-\hat{j}+5\hat{k}) = \vec{0}$

$\Rightarrow x + y + 2z = 0$

$2x - 3y - z = 0$

$-3x + 2y + 5z = 0$

Solving these equations, we get x = 0 = y = z

$\Rightarrow$ vectors are linearly independent.

Ans.[A]

Ex.13 If $\vec{a} = p\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b} = \sqrt{p}\,\hat{i}+\sqrt{13}\,\hat{j}$ are vectors of equal magnitude then p is equal to-

(A) 0 (B) 1

(C) 0 or 1 (D) 2

Sol. $|\vec{a}| = \sqrt{p^2+4+9} = \sqrt{p^2+13}$

$|\vec{b}| = \sqrt{p+13}$

$\therefore |\vec{a}| = |b| \Rightarrow p^2 + 13 = p + 13$

$\Rightarrow p = 0$ or 1 **Ans.[C]**

Ex.14 If $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ are position vectors of vertices of a triangle, then it is-

(A) equilateral

(B) isosceles

(C) right angled isosceles

(D) None of these

Sol. If given points are A, B, C respectively, then

$|\overrightarrow{AB}| = |-\hat{i}-2\hat{j}-6\hat{k}|$

$= \sqrt{1+4+36} = \sqrt{41}$

$|\overrightarrow{BC}| = |2\hat{i}-\hat{j}+\hat{k}|$

$= \sqrt{4+1+1} = \sqrt{6}$

$|\overrightarrow{CA}| = |-\hat{i}+3\hat{j}+5\hat{k}|$

$= \sqrt{1+9+25} = \sqrt{35}$

Now $|\overrightarrow{AB}|^2 = |\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2$

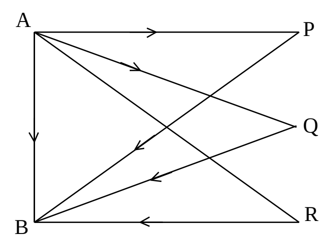
Hence given points form a right angled triangle. **Ans.[D]**

Ex.15 A, B, P, Q, R are five points in any plane. If forces $\overrightarrow{AP}$, $\overrightarrow{AQ}$, $\overrightarrow{AR}$ acts on point A and force $\overrightarrow{PB}$, $\overrightarrow{QB}$, $\overrightarrow{RB}$ acts on point B then resultant is-

(A) $3\,\overrightarrow{AB}$ (B) $3\,\overrightarrow{BA}$

(C) $3\,\overrightarrow{PQ}$ (D) $4\,\overrightarrow{PR}$

Sol. From figure



$\overrightarrow{AP} + \overrightarrow{PB} = \overrightarrow{AB}$

$\overrightarrow{AQ} + \overrightarrow{QB} = \overrightarrow{AB}$

$\overrightarrow{AR} + \overrightarrow{RB} = \overrightarrow{AB}$

So$(\overrightarrow{AP}+\overrightarrow{AQ}+\overrightarrow{AR}) + (\overrightarrow{PB}+\overrightarrow{QB}+\overrightarrow{RB}) = 3\overrightarrow{AB}$

so required resultant = $3\,\overrightarrow{AB}$. **Ans.[A]**

Ex.16 The length of diagonals of parallelogram with adjacent sides as $2\hat{i}+3\hat{j}-2\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$ are-

(A) $\sqrt{35}$, $\sqrt{35}$ (B) $\sqrt{11}$, $\sqrt{11}$

(C) $\sqrt{35}$, $\sqrt{11}$ (D) None of these

Sol. Let the given vectors be $\vec{a}$ and $\vec{b}$, then the diagonals will be represented by $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$. So the diagonals are represented by $3\hat{i}+5\hat{j}-\hat{k}$ and $\hat{i}+\hat{j}-3\hat{k}$.

Hence their lengths are

$= \sqrt{9+25+1}$ and $\sqrt{1+1+9}$

$= \sqrt{35}, \sqrt{11}$ **Ans.[C]**

Ex.17 If two vertices of a triangle are respectively $\hat{i}-\hat{j}$ and $\hat{j}+\hat{k}$, then the third vertex may be-

(A) $\hat{i}+\hat{k}$ (B) $\hat{i}-\hat{k}$

(C) $2\hat{i}-\hat{j}$ (D) All three

Sol. In the given alternatives no vector is collinear with the $\hat{i}-\hat{j}$ and $\hat{j}+\hat{k}$. **Ans.[D]**

Ex.18 If $\vec{a} = 2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}+3\hat{k}$, then $|\vec{a}+\vec{b}|$ is equal to-

(A) $4\sqrt{6}$ (B) $3\sqrt{6}$

(C) $2\sqrt{6}$ (D) $\sqrt{6}$

Sol. $\vec{a}+\vec{b} = 3\hat{i}+3\hat{j}+6\hat{k}$

$\Rightarrow |\vec{a}+\vec{b}| = \sqrt{9+9+36} = \sqrt{54} = 3\sqrt{6}$

Ans.[B]

Ex.19 If A (4, 7, 8) , B = (2, 3, 4) and C = (2, 5,7) are vertices of a triangle ABC, then the length of bisector of angle A is -

(A) $\frac{3\sqrt{34}}{2}$ (B) $\frac{2\sqrt{34}}{3}$

(C) $\frac{\sqrt{34}}{2}$ (D) $\frac{\sqrt{34}}{3}$

Sol. If AD is bisector of angle A, then D divides BC in the ratio AB : AC

$$\therefore \overrightarrow{AD} = \frac{|\overrightarrow{AC}|\,\overrightarrow{AB} + |\overrightarrow{AB}|\,\overrightarrow{AC}}{|\overrightarrow{AB}| + |\overrightarrow{AC}|}$$

But $\overrightarrow{AB} = -2\hat{i}-4\hat{j}-4\hat{k}$

and $\overrightarrow{AC} = -2\hat{i}-2\hat{j}-\hat{k}$

$\Rightarrow |\overrightarrow{AB}| = 6,\ |\overrightarrow{AC}| = 3$

$\therefore \overrightarrow{AD} = \frac{2}{3}(-3i-4j-3k)$

$\Rightarrow |\overrightarrow{AD}| = \frac{2\sqrt{34}}{3}$ **Ans.[B]**

Ex.20 The vector $\vec{c}$, directed along the internal bisector of the angle between the vectors $7\hat{i}-4\hat{j}-4\hat{k}$ and $-2\hat{i}-\hat{j}+2\hat{k}$ with $|\vec{c}| = 5\sqrt{6}$ is-

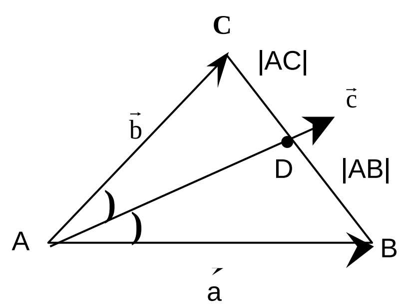
(A) $\frac{5}{3}\ \hat{i}-7\hat{j}+2\hat{k}$ (B) $\frac{5}{3}(5\hat{i}+5\hat{j}+2\hat{k})$

(C) $\frac{5}{3}(\hat{i}+7\hat{j}+2\hat{k})$ (D) None of these

Sol. Let $\vec{a} = 7\hat{i}-4\hat{j}-4\hat{k}$

and $\vec{b} = -2\hat{i}-\hat{j}+2\hat{k}$

Internal bisector divides the BC in the ratio of



$|\overrightarrow{AB}| : |\overrightarrow{AC}|,\ |\overrightarrow{AB}| = 9,\ |\overrightarrow{AC}| = 3$

$$\overrightarrow{AD} = \left(\frac{9(-2\hat{i}-\hat{j}+2\hat{k}) + 3(7\hat{i}-4\hat{j}-4\hat{k})}{9+3}\right)$$

$$\overrightarrow{AD} = \frac{\hat{i}-7\hat{j}+2\hat{k}}{4}$$

$$\vec{C} = \left(\frac{\overrightarrow{AD}}{|\overrightarrow{AD}|}\right) \pm 5\sqrt{6}$$

$= \pm\frac{5}{3}(\hat{i}-7\hat{j}+2\hat{k})$ **Ans.[A]**

Ex.21 The projection of the vector $\hat{i}+\hat{j}+\hat{k}$ on vector $\hat{i}+2\hat{j}+3\hat{k}$ is-

(A) $\frac{1}{\sqrt{14}}$ (B) $\frac{3}{\sqrt{14}}$

(C) $\frac{6}{\sqrt{14}}$ (D) $\frac{2}{3}$

Sol. Let $\vec{a} = \hat{i}+\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+2\hat{j}+3\hat{k}$ and so projection of $\vec{a}$ on $\vec{b}$ is

$$= \frac{(\hat{i}+\hat{j}+\hat{k}).(\hat{i}+2\hat{j}+3\hat{k})}{\sqrt{1+4+9}}$$

$= \frac{1+2+3}{\sqrt{1+4+9}} = \frac{6}{\sqrt{14}}$ **Ans.[C]**

Ex.22 If for three vectors $\vec{a}, \vec{b}, \vec{c}$; $\vec{a}.\vec{b} = \vec{a}.\vec{c}$; then which of the following is correct-

(A) $\vec{b} = \vec{c}$

(B) $\vec{a} = \vec{0}$

(C) $\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b} - \vec{c})$

(D) $\vec{a} \perp (\vec{b} - \vec{c})$

Sol. $\vec{a}.\vec{b} = \vec{a}.\vec{c}$

$\Rightarrow \vec{a}.(\vec{b} - \vec{c}) = 0$

$\Rightarrow \vec{a} = \vec{0}$ or $\vec{b} - \vec{c} = \vec{0}$ or $\vec{a} \perp (\vec{b} - \vec{c})$

$\Rightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b} - \vec{c})$

Ans.[C]

Ex.23 If moduli of vectors $\vec{a}, \vec{b}, \vec{c}$ are 3, 4 and 5 respectively $\vec{a}$ and $\vec{b}+\vec{c}$, $\vec{b}$ and $\vec{c}, \vec{c}+\vec{a}$ and $\vec{a}+\vec{b}$ are perpendicular to each other, then modulus of $\vec{a}+\vec{b}+\vec{c}$ is -

(A) $5\sqrt{2}$ (B) $2\sqrt{2}$ (C) 50 (D) 20

Sol. $\because \vec{a} \perp (\vec{b}+\vec{c}) \Rightarrow \vec{a}.\vec{b} + \vec{a}.\vec{c} = 0$

Similarly $\vec{b} \perp (\vec{c}+\vec{a}) \Rightarrow \vec{b}.\vec{c} + \vec{b}.\vec{a} = 0$

and $\vec{c} \perp (\vec{a}+\vec{b}) \Rightarrow \vec{c}.\vec{a} + \vec{c}.\vec{b} = 0$

$\therefore \vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} = 0$

Now

$|\vec{a}+\vec{b}+\vec{c}|^2$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a})$

$= 9 + 16 + 25 = 50$

$\therefore |\vec{a}+\vec{b}+\vec{c}| = 5\sqrt{2}$ **Ans.[A]**

Ex. 24 If θ be the angle between vectors $\vec{a} = \hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b} = 3\hat{i}+2\hat{j}+\hat{k}$, then $\cos\theta$ equals-

(A) 5/7 (B) 6/7

(C) 4/7 (D) 1/2

Sol. $\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{3+4+3}{\sqrt{14}\sqrt{14}} = 5/7$ **Ans.[A]**

Ex.25 If $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$ then angle between a and b is

(A) 60º (B) 30º

(C) 90º (D) 180º

Sol. $|a + b| = |\vec{a}-\vec{b}|$

$\Rightarrow |\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}.\vec{b}$

$\Rightarrow 4\vec{a}.\vec{b} = 0$

$\Rightarrow \vec{a}.\vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$ **Ans.[C]**

Ex. 26 Forces $3\hat{i}+2\hat{j}+5\hat{k}$ and $2\hat{i}+\hat{j}+3\hat{k}$ are acting at a particle which is displaced from point $2\hat{i}-\hat{j}-3\hat{k}$ to the point $4\hat{i}-3\hat{j}+\hat{k}$. The work done by forces is-

(A) 30 units (B) 36 units

(C) 24 units (D) 18 units

Sol. Resultant force

$\vec{F} = (3\hat{i}+2\hat{j}+5\hat{k}) + (2\hat{i}+\hat{j}+3\hat{k})$

$= 5\hat{i}+3\hat{j}+8\hat{k}$

Displacement vector

$\vec{d} = (4\hat{i}-3\hat{j}+\hat{k}) - (2\hat{i}-\hat{j}-3\hat{k})$

$= 2\hat{i}-2\hat{j}+4\hat{k}$

$\therefore$ work done by force = $\vec{F}.\vec{d}$

= 10 – 6 + 32 = 36 units

Ans.[B]

Ex.27 For any two vectors $\vec{a}$ and $\vec{b}$.$|\vec{a}\times b|^2$ equals-

(A) $|\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2$ (B) $|\vec{a}|^2 + |\vec{b}|^2$

(C) $|\vec{a}|^2 - |\vec{b}|^2$ (D) None of these

Sol. $|\vec{a}\times\vec{b}|^2 = (|\vec{a}||\vec{b}|\sin\theta)^2$

$= |\vec{a}|^2|\vec{b}|^2\sin^2\theta = |\vec{a}|^2|\vec{b}|^2(1-\cos^2\theta)$

$= |\vec{a}|^2|\vec{b}|^2 - (|\vec{a}||\vec{b}|\cos\theta)^2$

$= |\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2$ **Ans.[A]**

Ex.28 If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a}+\vec{b}+\vec{c} = \vec{0}$, then-

(A) $\vec{a}\times\vec{b} = \vec{b}\times\vec{c}$

(B) $\vec{b}\times\vec{c} = \vec{c}\times\vec{a}$

(C) $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$

(D) None of these

Sol. $\because \vec{a}+\vec{b}+\vec{c} = \vec{0} \Rightarrow \vec{c} = -(\vec{a}+\vec{b})$

$\therefore \vec{b}\times\vec{c} = -\vec{b}\times(\vec{a}+\vec{b})$

$= -\vec{b}\times\vec{a} - \vec{b}\times\vec{b}$

$= \vec{a}\times\vec{b}$

Similarly $\vec{c}\times\vec{a} = \vec{a}\times\vec{b}$

$\therefore\ \vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$ **Ans.[C]**

Ex.29 If $\ell\hat{i}+m\hat{j}+n\hat{k}$ is a unit vector which is perpendicular to vectors $2\hat{i}-\hat{j}+\hat{k}$ and $3\hat{i}+4\hat{j}-\hat{k}$ then $|\ell|$ is equal to-

(A) $-\frac{3}{\sqrt{155}}$ (B) $\sqrt{\frac{3}{155}}$

(C) $\frac{3}{\sqrt{155}}$ (D) None of these

Sol. Vector $2\hat{i}-\hat{j}+\hat{k}$ and $3\hat{i}+4\hat{j}-\hat{k}$

$$= \frac{(2\hat{i}-\hat{j}+\hat{k})\times(3\hat{i}+4\hat{j}-\hat{k})}{|(2\hat{i}-\hat{j}+\hat{k})\times(3\hat{i}+4\hat{j}-\hat{k})|}$$

$$= \frac{\hat{i}(1-4)-\hat{j}(-2-3)+\hat{k}(8+3)}{\sqrt{9+25+121}}$$

$$= \frac{-3\hat{i}+5\hat{j}+11\hat{k}}{\sqrt{155}}$$

$$\therefore |\ell| = \left|\frac{-3}{\sqrt{155}}\right| = \frac{3}{\sqrt{155}}$$ **Ans.[C]**

Ex.30 The unit vector perpendicular to the plane passing through points P($\hat{i}-\hat{j}+2\hat{k}$), Q($2\hat{i}-\hat{k}$) and R($2\hat{j}+\hat{k}$) is-

(A) $2\hat{i}+\hat{j}+\hat{k}$ (B) $\sqrt{6}\,(2\hat{i}+\hat{j}+\hat{k})$

(C) $\frac{1}{\sqrt{6}}\,(2\hat{i}+\hat{j}+\hat{k})$ (D) $\frac{1}{6}\,(2\hat{i}+\hat{j}+\hat{k})$

Sol. $\overrightarrow{PQ} = (2\hat{i}-\hat{k}) - (\hat{i}-\hat{j}+2\hat{k}) = \hat{i}+\hat{j}-3\hat{k}$

$\overrightarrow{PR} = (2\hat{i}+\hat{k}) - (\hat{i}-\hat{j}+2\hat{k}) = -\hat{i}+3\hat{j}-\hat{k}$

Now $\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & -3\\ -1 & 3 & -1\end{vmatrix}$

$= 8\hat{i}+4\hat{j}+4\hat{k}$

$\Rightarrow |\overrightarrow{PQ}\times\overrightarrow{PR}| = \sqrt{64+16+16} = 4\sqrt{6}$

$\therefore$ reqd. unit vector $= \frac{4(2\hat{i}+\hat{j}+\hat{k})}{4\sqrt{6}}$

$= \frac{1}{\sqrt{6}}(2\hat{i}+\hat{j}+\hat{k})$ **Ans.[C]**

Ex.31 If $\vec{a},\vec{b},\vec{c}$ be three non- zero vectors, then $|(\vec{a}\times\vec{b}).\ \vec{c}\ | = |\vec{a}\,||\vec{b}\,||\vec{c}\,|$, if -

(A) $\vec{a}.\vec{b} = \vec{b}.\vec{c} = 0$

(B) $\vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$

(C) $\vec{c}.\vec{a} = \vec{a}.\vec{b} = 0$

(D) $\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$

Sol. $|(\vec{a}\times\vec{b}).\ c| = |\vec{a}\,||\vec{b}\,||\vec{c}\,|$

$\Leftrightarrow |(a\times\vec{b})\,||\vec{c}\,|\cos\theta = |\vec{a}\,||\vec{b}\,||\vec{c}\,|$

(where θ is the angle between $\vec{a}\times\vec{b}$ and $\vec{c}$)

$\Leftrightarrow |\vec{a}\,||\vec{b}\,||\vec{c}\,|\sin\phi\cos\theta = |\vec{a}\,||\vec{b}\,||\vec{c}\,|$

(where ϕ is the angle between $\vec{a}$ and $\vec{b}$)

$\Leftrightarrow \sin\phi = 1,\ \cos\theta = 1$

$\Leftrightarrow \phi = 90^{\circ},\ \theta = 0^{\circ}$

$\Leftrightarrow \vec{a}.\vec{b} = 0,\ \vec{a}.\vec{c} = 0,\ \vec{b}.\vec{c} = 0$ **Ans.[D]**

Ex.32 If $\vec{a}.\vec{b} = \vec{a}.\vec{c}$ and $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$, then correct statement is-

(A) $\vec{a}\parallel(\vec{b}-\vec{c})$ (B) $\vec{a}\,(\vec{b}-\vec{c})$

(C) $\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ (D) None of these

Sol. $\vec{a}.\vec{b} = \vec{a}.\vec{c} \Rightarrow \vec{a}.(\vec{b}-\vec{c}) = 0$

$\Rightarrow \vec{a} = \vec{0}$ or $\vec{b}-\vec{c} = \vec{0}$ or $\vec{a}\perp(\vec{b}-\vec{c})$

$\Rightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a}\perp(\vec{b}-\vec{c})$ ...(1)

Also $\vec{a}\times\vec{b} = \vec{a}\times\vec{c} \Rightarrow \vec{a}\times(\vec{b}-\vec{c}) = \vec{0}$

$\Rightarrow \vec{a} = \vec{0}$ or $\vec{b}-\vec{c} = \vec{0}$ or $\vec{a}\parallel(\vec{b}-\vec{c})$

$\Rightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a}\parallel(\vec{b}-\vec{c})$ ...(2)

Observing to (1) and (2) we find that

$\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ **Ans.[C]**

Ex.33 The components of any vector a along and perpendicular to a non-zero vector b are-

(A) $\frac{\vec{a}.\vec{b}}{|\vec{b}\,|},\ \frac{\vec{a}.\vec{b}}{|\vec{a}\,|}$ (B) $\frac{|\vec{a}\times\vec{b}\,|}{|\vec{b}\,|},\ \frac{|\vec{a}\times\vec{b}\,|}{|\vec{b}\,|}$

(C) $\frac{\vec{a}.\vec{b}}{|\vec{b}\,|},\ \frac{|\vec{a}\times\vec{b}\,|}{|\vec{b}\,|}$ (D) $\frac{\vec{a}.\vec{b}}{|\vec{b}\,|},\ \frac{|\vec{a}\times\vec{b}\,|}{|\vec{a}\,|}$

Sol. Let θ be the angle between $\vec{a}$ and $\vec{b}$, then component of $\vec{a}$ along $\vec{b}$

$$= |\vec{a}\,|\cos\theta = \frac{|\vec{a}\,|\,\vec{a}.\vec{b}}{|\vec{a}\,|\,|\vec{b}\,|} = \frac{\vec{a}.\vec{b}}{|\vec{b}\,|}$$

and component of $\vec{a}$ perpendicular to $\vec{b}$

$$= |\vec{a}\,|\sin\theta = \frac{|\vec{a}\,||\vec{a}\times\vec{b}\,|}{|\vec{a}\,|\,|\vec{b}\,|}$$

$$= \frac{|\vec{a}\times\vec{b}\,|}{|\vec{b}\,|}$$ **Ans.[C]**

Ex.34 The area of parallelogram whose diagonals are respectively $3\hat{i}+\hat{j}-2\hat{k}$ and $\hat{i}-3\hat{j}+4\hat{k}$ is-

(A) $5\sqrt{2}$ (B) $5\sqrt{3}$ (C) $2\sqrt{5}$ (D) $3\sqrt{5}$

Sol. Area of parallelogram = $\frac{1}{2}|\vec{a}\times\vec{b}|$

where $\vec{a} = 3\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b} = \hat{i}-3\hat{j}+4\hat{k}$

now $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$

$= -2\hat{i}-14\hat{j}-10\hat{k}$

Area of parallelogram

$= \frac{1}{2}|-2\hat{i}-14\hat{j}-10\hat{k}|$

$= \sqrt{1+49+25} = 5\sqrt{3}$ **Ans.[B]**

Ex.35 If $\hat{i}-\hat{j}+2\hat{k}$, $2\hat{i}+\hat{j}-\hat{k}$ and $3\hat{i}-\hat{j}+2\hat{k}$ are position vectors of vertices of a triangle, then its area is-

(A) 26 (B) 13 (C) $2\sqrt{13}$ (D) $\sqrt{13}$

Sol. If A, B, C are given vertices, then

$\overrightarrow{AB} = \hat{i}+2\hat{j}-3\hat{k}$, $\overrightarrow{AC} = 2\hat{i}$

$\therefore \overrightarrow{AB}\times\overrightarrow{AC} = (\hat{i}+2\hat{j}-3\hat{k})\times 2\hat{i} = -4\hat{k}-6\hat{j}$

$\Rightarrow \left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \sqrt{16+36} = 2\sqrt{13}$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \sqrt{13}$

Ans. [D]

Ex.36 If A,B,C,D are any four points, then $|\overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}|$ equals-

(A) Area of ΔABC

(B) 2(Area of ΔABC)

(C) 3(Area of ΔABC)

(D) 4 (Area of ΔABC)

Sol. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ be position vectors of points A,B,C and D respectively, then

$\overrightarrow{AB}\times\overrightarrow{CD} = (\vec{b}-\vec{a})\times(\vec{d}-\vec{c})$

$= \vec{b}\times\vec{d} - \vec{b}\times\vec{c} - \vec{a}\times\vec{d} + \vec{a}\times\vec{c}$

Similarly

$\overrightarrow{BC}\times\overrightarrow{AD}$

$= \vec{c}\times\vec{d} - \vec{c}\times\vec{a} - \vec{b}\times\vec{d} + \vec{b}\times\vec{a}$

$\overrightarrow{CA}\times\overrightarrow{BD}$

$= \vec{a}\times\vec{d} - \vec{a}\times\vec{b} - \vec{c}\times\vec{d} + \vec{c}\times\vec{b}$

Therefore given expression

$= |2(\vec{b}\times\vec{a} - \vec{b}\times\vec{c} + \vec{a}\times\vec{c})|$

$= 2|(\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a})|$

= 4 (Area of ΔABC) **Ans.[D]**

Ex. 37 $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are the position vectors of four coplanar points A, B, C and D respectively. If $(\vec{a}-\vec{d}).(\vec{b}-\vec{c}) = 0$

$= (\vec{b}-\vec{d}).(\vec{c}-\vec{a})$, then for the ΔABC,D is-

(A) incentre (B) orthocentre

(C) circumcentre (D) centroid

Sol. $(\vec{b}-\vec{d}).(\vec{c}-\vec{a}) = 0$

$\Rightarrow (\vec{a}-\vec{d}) \perp (\vec{b}-\vec{c}) \Rightarrow \overrightarrow{AD} \perp \overrightarrow{BC}$

Similarly $(\vec{b}-\vec{d}).(\vec{c}-\vec{a}) = 0$

$\Rightarrow \overrightarrow{BD} \perp \overrightarrow{AC}$

$\therefore$ D is the orthocentre of ΔABC.

Ans.[B]

Ex.38 Force $\hat{i}+2\hat{j}-3\hat{k}$, $2\hat{i}+3\hat{j}+4\hat{k}$ and $-\hat{i}-\hat{j}+\hat{k}$ are acting at the point P (0, 1, 2). The moment of these forces about the point A (1, –2, 0) is-

(A) $2\hat{i}-6\hat{j}+10\hat{k}$ (B) $-2\hat{i}+6\hat{j}-10\hat{k}$

(C) $2\hat{i}+6\hat{j}-10\hat{k}$ (D) None of these

Sol. If $\vec{F}$ be the resultant force, then

$\vec{F} = 2\hat{i}+4\hat{j}+2\hat{k}$

Also $\vec{r} = \overrightarrow{AP} = -\hat{i}+3\hat{j}+2\hat{k}$

$\therefore$ required moment = $\vec{r}\times\vec{F}$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 3 & 2 \\ 2 & 4 & 2 \end{vmatrix} = -2\hat{i}+6\hat{j}-10\hat{k}$ **Ans.[B]**

Ex.39 $\vec{a}.(\vec{b}+\vec{c})\times(\vec{a}+\vec{b}+\vec{c})$ is equal to-

(A) 0 (B) 2 [abc]

(C) [abc] (D) None of these

Sol. $\vec{a}.(\vec{b}+\vec{c})\times(\vec{a}+\vec{b}+\vec{c})$

$= \vec{a}.[(\vec{b}+\vec{c})\times\vec{a}+(\vec{b}+\vec{c})\times\vec{b}+(\vec{b}+\vec{c})\times\vec{c}]$

$= \vec{a}.[(\vec{b}\times\vec{a}+\vec{c}\times\vec{a}+\vec{b}\times\vec{b}+\vec{c}\times\vec{c}\times\vec{b}+\vec{c}\times\vec{c}]$

$= \vec{a}.[\vec{b}\times\vec{a}+\vec{c}\times\vec{a}]$

$= [\vec{a}\ \vec{b}\ \vec{c}]+[\vec{a}\ \vec{c}\ \vec{a}]$

$= 0$ **Ans.[D]**

Ex.40 If vectors $\vec{a}=\hat{i}-\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=3\hat{i}+p\hat{j}+5\hat{k}$ are coplanar, then the value of p is-

(A) 2 (B) 6

(C) – 2 (D) – 6

Sol. $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar $\Rightarrow$ $[\vec{a}\ \ \vec{b}\ \ \vec{c}]=0$

$$\Rightarrow \begin{vmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 3 & p & 5 \end{vmatrix} = 0$$

$\Rightarrow (10 + p + 3) – (6 – 5 – p) = 0$

$\Rightarrow$ p = – 6 **Ans.[D]**

Ex.41 If $\vec{a}=-3\hat{i}+8\hat{j}+5\hat{k}$, $\vec{b}=-3\hat{i}+7\hat{j}-3\hat{k}$ and $\vec{c}=7\hat{i}-5\hat{j}-3\hat{k}$ are the coterminus edges of a parallelopiped then its volume is-

(A) 108 (B) 210

(C) 272 (D) 308

Sol. Required volume = $[\vec{a}\ \ \vec{b}\ \ \vec{c}]$

$$= \begin{vmatrix} -3 & 7 & 5 \\ -3 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix}$$

$= | – 3 (– 21 – 15) – 7 (9 + 21) + 5 (15 – 49)]$

$= |108 – 210 – 170|$

$= 272$ **Ans.[C]**

Ex.42 For any vector $\vec{a}$, $\vec{u} = \hat{i}\times(\vec{a}\times\hat{i}) + \hat{j}\times(\vec{a}\times\hat{j}) + \hat{k}\times(\vec{a}\times\hat{k})$ equals-

(A) 2 $\vec{a}$ (B) – 2 $\vec{a}$

(C) $\vec{a}$ (D) – $\vec{a}$

Sol. $\vec{u} = (\hat{i}.\hat{i})\vec{a} - (\hat{i}.\vec{a})\hat{i} + (\hat{j}.\hat{j})\vec{a} - (\hat{j}.\vec{a})\hat{j} + (\hat{k}.\hat{k})\vec{a} - (\hat{k}.\vec{a})\hat{k}$

$= \vec{a} - a_1\hat{i} + \vec{a} - a_2\hat{j} + \vec{a} - a_3\hat{k}$

$[\vec{a} = a_1 i + a_2\hat{j} + a_3\hat{k}$ (say)$]$

$\therefore$ u = $3\vec{a} - \vec{a} = 2\vec{a}$ **Ans.[A]**

Ex.43 Let $\vec{a}, \vec{b}, \vec{c}$ such that $|\vec{a}| = 1$, $|\vec{b}| = 1$ and $|\vec{c}| = 2$ and if $\vec{a}\times(\vec{a}\times\vec{c}) + \vec{b} = \vec{0}$ then acute angle between $\vec{a}$ and $\vec{c}$ is -

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{6}$ (D) None of these

Sol. If angle between $\vec{a}$ and $\vec{c}$ is θ then –

$\vec{a}.\vec{c} = |\vec{a}||\vec{c}|\cos\theta$

$= 1.2\cos\theta = 2\cos\theta$

but $\vec{a}\times(\vec{a}\times\vec{c})+\vec{b} = \vec{0}$

$\Rightarrow (\vec{a}.\vec{c})\vec{a} - (\vec{a}.\vec{a})\vec{c} + \vec{b} = \vec{0}$

$\Rightarrow (2\cos\theta)\vec{a} - 1.\vec{c} = -\vec{b}$

$\Rightarrow [(2\cos\theta)\vec{a} - \vec{c}]^2 = [-\vec{b}]^2$

$\Rightarrow 4\cos^2\theta|\vec{a}|^2 - 2.(2\cos\theta)\vec{a}.\vec{c} + |\vec{c}|^2 = |\vec{b}|^2$

$\Rightarrow 4\cos^2\theta - 4\cos\theta(2\cos\theta) + 4 = 1$

$\Rightarrow 4(1 - \cos^2\theta) = 1[\because |\vec{a}| = 1, |\vec{b}| = 1]$

$\Rightarrow \sin\theta = 1/2$

$\Rightarrow \theta = \frac{\pi}{6}$ **Ans.[C]**

Ex.44 Let $\vec{a} = 2\hat{i}-\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+2\hat{j}-\hat{k}$ & $\vec{c} = \hat{i}+\hat{j}=2\hat{k}$ be three vectors. A vector in the plane of b and c whose projection on a is $\sqrt{2/3}$ will be-

(A) $2\hat{i}+3\hat{j}-3\hat{k}$ (B) $2\hat{i}+3\hat{j}+3\hat{k}$

(C) $-2\hat{i}-\hat{j}+5\hat{k}$ (D) $2\hat{i}+\hat{j}+5\hat{k}$

Sol. Let the required vector $\vec{r} = \vec{b}+t\vec{c}$

$\Rightarrow \vec{r} = (1 + t)\hat{i} + (2 + t)\hat{j} - (1 + 2t)\hat{k}$

Also projection of $\vec{r}$ on $\vec{a} = \sqrt{2/3}$

$\Rightarrow \frac{\vec{r}.\vec{a}}{|\vec{a}|} = \sqrt{2/3}$

$= \frac{2(1+t)-(2+t)-(1+2t)}{\sqrt{6}} = \sqrt{\frac{2}{3}}$

$\Rightarrow – t – 1 = 2$

$\Rightarrow$ t = – 3

$\therefore \vec{r} = -2\hat{i}-\hat{j}+5\hat{k}$ **Ans.[C]**

Ex.45 If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and $\vec{A} = (1,a,a^2)$, $\vec{B} = (1, b, b^2)$ and $\vec{C} = (1, c, c^2)$ are non-coplanar vectors, then (abc) equals-

(A) 0 (B) 1
(C) – 1 (D) 2

Sol. Since $\vec{A}, \vec{B}, \vec{C}$ are non-coplanar vector

$\therefore \Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0$

Now $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$

$\Rightarrow (abc+1) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

$\Rightarrow abc + 1 = 0$ $[\because \Delta \neq 0]$

$\therefore abc = -1$ **Ans.[C]**

Ex.46 A unit vector in xy- plane which makes 45º angle with vector $\hat{i}+\hat{j}$ and 60º angle with vector 3i – 4j will be-

(A) $\hat{i}$ (B) $(\hat{i}+\hat{j})/\sqrt{2}$
(C) $(\hat{i}-\hat{j})/\sqrt{2}$ (D) None of these

Sol. Let the required vector be $\vec{r} = x\hat{i}+y\hat{j}$

$\therefore x^2 + y^2 = 1$ ...(1)

If given vectors be a and b respectively, then as given

$\cos 45^\circ = \frac{\vec{r}.\vec{a}}{|\vec{r}||\vec{a}|} \Rightarrow x = y = 1$ ...(2)

$\cos 60^\circ = \frac{\vec{r}.\vec{b}}{|\vec{r}||\vec{b}|} \Rightarrow 6x - 8y = 5$ ...(3)

But (1), (2) and (3) do not hold together. Hence such a vector is not possible.

Ans.[D]

Ex.47 If vectors $a\hat{i}+\hat{j}+\hat{k}$, $\hat{i}+b\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+c\hat{k}$ $(a \neq b \neq c \neq 1)$ are coplanar, then $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$ equals-

(A) 1 (B) 0
(C) – 1 (D) None of these

Sol. Since vectors are coplanar,

$\therefore \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} a & 1 & 1 \\ 1-a & b-1 & 0 \\ 0 & 1-b & c-1 \end{vmatrix} = 0$

[Using $R_2 - R_1$, $R_3 - R_2$]

$\Rightarrow a(b-1)(c-1) - (1-a)\{(c-1) - (1-b)\} = 0$

$\Rightarrow a(1-b)(1-c) + (1-a)(1-c) + (1-a)(1-b) = 0$

$\Rightarrow (a-1+1)(1-b)(1-c) + (1-a)(1-c) + (1-a)(1-b) = 0$

$\Rightarrow (1-b)(1-c) + (1-a)(1-c) + (1-a)(1-b) = (1-a)(1-b)(1-c)$

$\Rightarrow \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$ **Ans.[A]**

Ex.48 Let $\vec{a} = 2\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b} = \hat{i}+\hat{j}$ if vector $\vec{c}$ is such that $\vec{a}.\vec{c} = |\vec{c}|$, $|\vec{c}-\vec{a}| = 2\sqrt{2}$ and angle between $(\vec{a} \times \vec{b})$ and $\vec{c}$ is the 30º then $|(\vec{a}\times\vec{b})\times\vec{c}|$ is equal to -

(A) $\frac{2}{3}$ (B) $\frac{3}{2}$ (C) 2 (D) 3

Sol. $|\vec{c}-\vec{a}|^2 = (\vec{c}-\vec{a}).(\vec{c}-\vec{a}) = (2\sqrt{2})^2$

$\Rightarrow |\vec{c}|^2 + |\vec{a}|^2 - 2\vec{c}.\vec{a} = 8$

$\Rightarrow |\vec{c}|^2 + (4+1+4) - 2\vec{c}.\vec{a} = 8$

$\Rightarrow |\vec{c}|^2 + 9 - 2|\vec{c}| = 8$ $[\because \vec{a}.\vec{c} = |\vec{c}|]$

$\Rightarrow |\vec{c}|^2 - 2|\vec{c}| + 1 = 0$

$|\vec{c}-1|^2 = 0 \Rightarrow \vec{c} = 1$

$|(\vec{a}\times\vec{b})\times\vec{c}| = |\vec{a}\times\vec{b}||\vec{c}| \sin 30^\circ$

$= 1 \times \frac{1}{2} |\vec{a}\times\vec{b}| = \frac{1}{2}|\vec{a}\times\vec{b}|$

But $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\hat{i}-2\hat{j}+\hat{k}$

$\therefore |\vec{a}\times\vec{b}| = \sqrt{4+4+1} = 3$

$\therefore |(\vec{a}\times\vec{b})\times\vec{c}| = \frac{3}{2}$ **Ans.[B]**