

# 3-D COORDINATE GEOMETRY

(KEY CONCEPTS + SOLVED EXAMPLES)

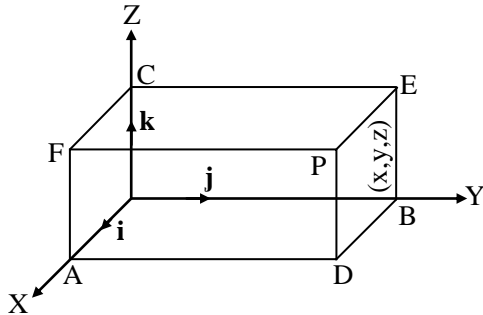
## —3-D COORDINATE GEOMETRY—

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## KEY CONCEPT

### 1. Coordinates of a point in space

Let O be a fixed point known as origin and let OX, OY and OZ be three mutually perpendicular lines, taken as x-axis, y-axis and z-axis respectively in such a way that they form a right-handed system.

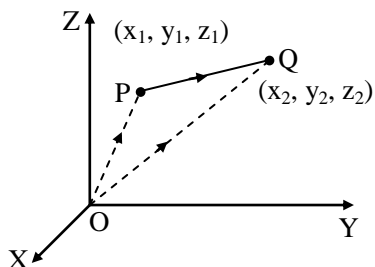


The planes XOY, YOZ and ZOX are known as xy-plane, yz-plane and zx-plane respectively.

Let P be a point in space and distances of P from yz, zx and xy-planes be x, y, z respectively (with proper signs), then we say that coordinates of P are (x, y, z). Also OA = x, OB = y, OC = z.

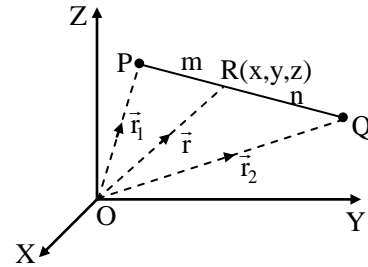
### 2. Distance between two points

If  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$  are two points, then distance between them  $PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$ . In particular distance of a point (x, y, z) from origin  $= \sqrt{x^2 + y^2 + z^2}$ .



### 3. Coordinates of division point

Coordinates of the point dividing the line joining two points  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$  in the ratio  $m_1 : m_2$  are



(i) In case of internal division

$$\left( \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}, \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2} \right)$$

(ii) In case of external division

$$\left( \frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}, \frac{m_1 z_2 - m_2 z_1}{m_1 - m_2} \right)$$

**Note :**

**(a) Coordinates of the Mid point :**

When division point is the midpoint of PQ, then ratio will be 1 : 1; hence coordinates of the midpoint of PQ are  $\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$

**(b) Centroid of a Triangle :**

If  $(x_1, y_1, z_1)$ ,  $(x_2, y_2, z_2)$  and  $(x_3, y_3, z_3)$  be the vertices of a triangle, then the centroid of the triangle is

$$\left( \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right)$$

**(c) Division by Coordinate Planes :**

The ratios in which the line segment PQ joining  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$  is divided by coordinate planes are as follows.

(i) by yz - plane:  $-\frac{x_1}{x_2}$  ratio

(ii) by zx - plane:  $-\frac{y_1}{y_2}$  ratio

(iii) by xy - plane :  $-\frac{z_1}{z_2}$  ratio

**(d) Centroid of a Tetrahedron :**

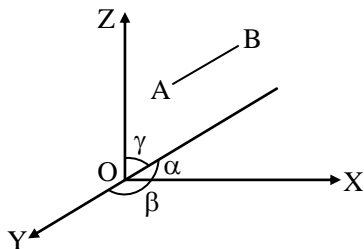
If  $(x_r, y_r, z_r)$ ,  $r = 1, 2, 3, 4$  are vertices of a tetrahedron, then coordinates of its centroid are  $\left( \frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4} \right)$

## 4. Direction cosines & direction ratio's of a line

### 4.1 Direction cosines of a line [Dc's] :

The cosines of the angles made by a line with coordinate axes are called the direction cosines of that line.

Let  $\alpha, \beta, \gamma$  be the angles made by a line AB with coordinate axes then  $\cos \alpha, \cos \beta, \cos \gamma$  are the direction cosines of AB which are generally denoted by  $l, m, n$ . Hence  $l = \cos \alpha, m = \cos \beta, n = \cos \gamma$



**Note :**

$-1 < \cos x < 1 \forall x \in \mathbb{R}$ , hence values of  $l, m, n$  are such real numbers which are not less than  $-1$  and not greater than  $1$ . Hence DC's  $\in [-1, 1]$

#### Direction cosines of coordinate axes :

x-axis makes  $0^\circ, 90^\circ$  and  $90^\circ$  angles with three coordinate axes, so its direction cosines are  $\cos 0^\circ, \cos 90^\circ, \cos 90^\circ$ , i.e.  $1, 0, 0$ .

Similarly direction cosines of y-axis and z-axis are  $0, 1, 0$  and  $0, 0, 1$  respectively. Hence

dc's of x - axis =  $1, 0, 0$

dc's of y - axis =  $0, 1, 0$

dc's of z - axis =  $0, 0, 1$

**Note :** (i) The direction cosines of a line parallel to any coordinates axis are equal to the direction cosines of the corresponding axis.

(ii) Relation between dc's :  $l^2 + m^2 + n^2 = 1$

### 4.2 Direction ratios of a line [DR's]

Three numbers which are proportional to the direction cosines of a line are called the direction ratios of that line. If  $a, b, c$  are such numbers which are proportional to the direction cosines  $l, m, n$  of a line then  $a, b, c$  are direction ratios of the line. Hence

$$a, b, c \text{ dr's} \Leftrightarrow \frac{a}{l} = \frac{b}{m} = \frac{c}{n}$$

Further we may observe that in above case

$$\begin{aligned} \frac{l}{a} = \frac{m}{b} = \frac{n}{c} &= \frac{1}{\sqrt{l^2 + m^2 + n^2}} \\ &= \pm \frac{1}{\sqrt{a^2 + b^2 + c^2}} \\ \Rightarrow l &= \pm \frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \pm \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \\ n &= \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

**Note :**

- Numbers of dr's are not unique whereas numbers of dc's are unique.
- $a^2 + b^2 + c^2 \neq 1$ .

### 4.3 Direction cosines of a line joining two points:

Let  $P \equiv (x_1, y_1, z_1)$  and  $Q \equiv (x_2, y_2, z_2)$ ; then

- dr's of PQ :  $(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)$
- dc's of PQ :  $\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$  i.e.

$$\frac{x_2 - x_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \frac{y_2 - y_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \frac{z_2 - z_1}{\sqrt{\Sigma(x_2 - x_1)^2}}$$

## 5. Angle between to lines

**Case-I :** When dc's of the lines are given

If  $l_1, m_1, n_1$  and  $l_2, m_2, n_2$  are dc's of given two lines, then the angle  $\theta$  between them is given by

$$* \cos \theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

$$* \sin \theta = \sqrt{(\ell_1 m_2 - \ell_2 m_1)^2 + (m_1 n_2 - m_2 n_1)^2 + (n_1 \ell_2 - n_2 \ell_1)^2}$$

The value of  $\sin \theta$  can easily be obtained by the following form :

$$\sin \theta = \sqrt{\left| \begin{array}{cc} \ell_1 & m_1 \\ \ell_2 & m_2 \end{array} \right|^2 + \left| \begin{array}{cc} m_1 & n_1 \\ m_2 & n_2 \end{array} \right|^2 + \left| \begin{array}{cc} n_1 & \ell_1 \\ n_2 & \ell_2 \end{array} \right|^2}$$

**Case-II :** When dr's of the lines are given

If  $a_1, b_1, c_1$  and  $a_2, b_2, c_2$  are dr's of given two lines, then the angle  $\theta$  between them is given by

$$* \cos \theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$* \sin \theta = \frac{\sqrt{\sum (a_1 b_2 - a_2 b_1)^2}}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

### Conditions of Parallelism and Perpendicularity of Two Lines :

If two lines are parallel then angle between them is  $0^\circ$  and if they are perpendicular then angle between them is  $90^\circ$ . In these cases using above formulae for  $\sin \theta$  and  $\cos \theta$  respectively, we shall get the following conditions.

**Case-I :** When dc's of two lines AB and CD, say  $l_1,$

$m_1, n_1$  and  $l_2, m_2, n_2$  are known

$$AB \parallel CD \Leftrightarrow l_1 = l_2, m_1 = m_2, n_1 = n_2$$

$$AB \perp CD \Leftrightarrow l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

**Case-II :** When dr's of two lines AB and CD, say  $a_1,$

$b_1, c_1$  and  $a_2, b_2, c_2$  are known

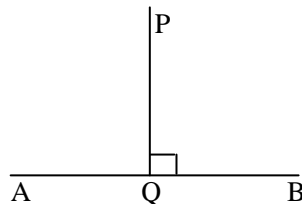
$$AB \parallel CD \Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$AB \perp CD \Leftrightarrow a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

### ➤ Projection of a point on a line

Let P be a point and AB be a given line. Draw perpendicular PQ from P on

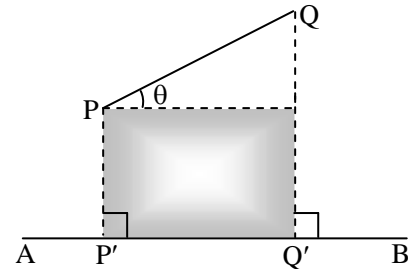
AB which meets it at Q. This point Q is called projection of P on the line AB.



### ➤ Projection of a line segment joining two points on a line

Let PQ be a line segment where  $P \equiv (x_1, y_1, z_1)$  and  $Q \equiv (x_2, y_2, z_2)$ ; and AB be a given line with dc's as  $l, m, n$ . If the line segment PQ makes  $\theta$  angle with the line AB, then projection of PQ is  $P'Q' = PQ \cos \theta$ . On replacing the value of  $\cos \theta$  in this, we shall get the following value of  $P'Q'$

$$P'Q' = l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$



**Note :**

(i) For x-axis,  $l = l, m = 0, n = 0$  hence

$$\begin{aligned} \text{Projection of PQ on x-axis} &= l(x_2 - x_1) + 0 + 0 \\ &= (x_2 - x_1) \end{aligned}$$

$$\text{Projection of PQ on y-axis} = y_2 - y_1$$

$$\text{Projection of PQ on z-axis} = z_2 - z_1$$

(ii)  $\therefore PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

= the sum of the squares of the projections of PQ on coordinate axes

$\therefore$  if  $a, b, c$  are the projections of a line segment on coordinate axes, then

$$\text{length of the segment} = \sqrt{a^2 + b^2 + c^2}$$

(iii) If  $a, b, c$  are projections of a line segment on coordinate axes then its dc's are

$$\pm \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \pm \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

### 6. Cartesian equation of a line passing through a given point & given direction ratios

Cartesian equation of a straight line passing through a fixed point  $(x_1, y_1, z_1)$  and having

direction ratios  $a, b, c$  is  $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

**Note**

(1) The parametric equations of the line  $\frac{x-x_1}{a} =$

$\frac{y-y_1}{b} = \frac{z-z_1}{c}$  are  $x = x_1 + a\lambda$ ,  $y = y_1 + b\lambda$ ,  
 $z = z_1 + c\lambda$ , where  $\lambda$  is the parameter.

(2) The coordinates of any point on the line  $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$  are

$(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$ , where  $\lambda \in \mathbb{R}$ .

(3) Since the direction cosines of a line are also direction ratios, therefore equation of a line passing through  $(x_1, y_1, z_1)$  and having direction cosines  $l, m, n$  is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

(4) Since  $x, y$  and  $z$ - axis passes through the origin and have direction cosines 1, 0, 0; 0, 1, 0 and 0, 0, 1 respectively. Therefore their equations are  $x$ -axis

$$\therefore \frac{x-0}{1} = \frac{y-0}{0} = \frac{z-0}{0} \text{ or } y = 0 \text{ and } z = 0$$

$$y\text{- axis : } \frac{x-0}{0} = \frac{y-0}{1} = \frac{z-0}{0} \text{ or } x = 0 \text{ and}$$

$$z = 0$$

$$z\text{- axis : } \frac{x-0}{0} = \frac{y-0}{0} = \frac{z-0}{1} \text{ or } x = 0 \text{ and}$$

$$y = 0$$

### CARTESIAN EQUATION OF A LINE PASSING THROUGH TWO GIVEN POINTS

The cartesian equation of a line passing through two given points  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  is given by

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

### Perpendicular distance of a point from a line

**Cartesian Form :** To find the perpendicular distance of a given point  $(\alpha, \beta, \gamma)$  from a given line

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

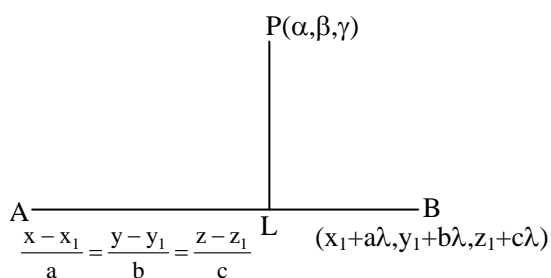
Let L be the foot of the perpendicular drawn from P  $(\alpha, \beta, \gamma)$  on the line  $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

Let the coordinates of L be  $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$ . Then direction ratios of PL are  $x_1 + a\lambda - \alpha, y_1 + b\lambda - \beta, z_1 + c\lambda - \gamma$ .

Direction ratio of AB are  $a, b, c$ . Since PL is perpendicular to AB, therefore

$$(x_1 + a\lambda - \alpha) a + (y_1 + b\lambda - \beta) b + (z_1 + c\lambda - \gamma) c = 0$$

$$\Rightarrow \lambda = \frac{a(\alpha - x_1) + b(\beta - y_1) + c(\gamma - z_1)}{a^2 + b^2 + c^2}$$



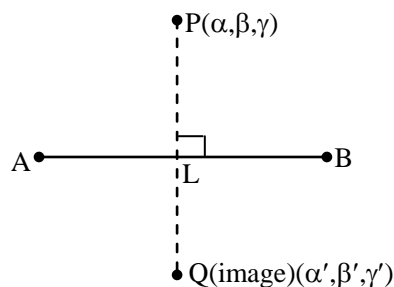
Putting this value of  $\lambda$  in  $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$ , we obtain coordinates of L. Now, using distance formula we can obtain the length PL.

### Reflection or image of a point in a straight line

**Cartesian form :** To find the reflection or image of a point in a straight line in cartesian form.

Let P  $(\alpha, \beta, \gamma)$  be the point and  $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$  be the given line

Let L be the foot of perpendicular from P to AB and let Q be the image of the point in the given line, where,  $PL = LQ$



Let the co-ordinate of L be ;

$$(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$$

Then direction ratios of PL are ;

$$(x_1 + a\lambda - \alpha, y_1 + b\lambda - \beta, z_1 + c\lambda - \gamma)$$

since PL is perpendicular to the given line, whose direction ratios are a, b, c

$$\therefore (x_1 + a\lambda - \alpha) \cdot a + (y_1 + b\lambda - \beta) \cdot b + (z_1 + c\lambda - \gamma) \cdot c = 0$$

$$\Rightarrow \lambda = \frac{\{a(\alpha - x_1) + b(\beta - y_1) + c(\gamma - z_1)\}}{a^2 + b^2 + c^2}$$

substituting  $\lambda$  we get L. (foot of perpendicular).

let co-ordinates of Q( $\alpha'$ ,  $\beta'$ ,  $\gamma'$ ) be image

$\therefore$  mid point of PQ is L.

$$\therefore \frac{\alpha + \alpha'}{2} = x_1 + a\lambda, \frac{\beta + \beta'}{2} = y_1 + b\lambda, \frac{\gamma + \gamma'}{2} = z_1 + c\lambda,$$

$$\therefore \alpha' = 2(x_1 + a\lambda) - \alpha, \beta' = 2(y_1 + b\lambda) - \beta, \gamma' = 2(z_1 + c\lambda) - \gamma$$

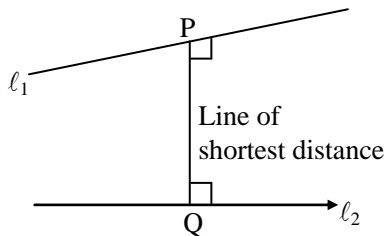
## 7. Skew lines

Two straight lines in a space which are neither parallel nor intersecting are called skew-lines.

Thus, the skew lines are those lines which do not lie in the same plane.

(i) **Shortest distance between two skew straight lines :** If  $l_1$  and  $l_2$  are two skew lines, then there is one and only one line perpendicular to each of lines  $l_1$  and  $l_2$  which is known as three line of shortest distance.

Here, distance PQ is called to be shortest distance.



**Vector form :**

Let  $l_1$  and  $l_2$  be two lines whose equations

are:  $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$  and  $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$  respectively

clearly  $l_1$  and  $l_2$  pass through the points A

and B with position vectors  $\vec{a}_1$  and  $\vec{a}_2$

respectively and are parallel to the vectors

$\vec{b}_1$  and  $\vec{b}_2$  respectively

$$\text{Distance } \overrightarrow{PQ} = \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}$$

**Condition for lines to intersect**

The two lines are intersecting if ;

$$\frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} = 0$$

$$\Rightarrow (\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = 0$$

$$\Rightarrow [b_1 b_2 (a_2 - a_1)] = 0$$

**Cartesian form :**

Let the two skew lines be :

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

$$\text{and } \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

$$\text{shortest distance} = \frac{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}$$

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(m_1 n_2 - m_2 n_1)^2 + (n_1 \ell_2 - n_2 \ell_1)^2 + (\ell_1 m_2 - \ell_2 m_1)^2}}$$

**Conditions for lines to intersect**

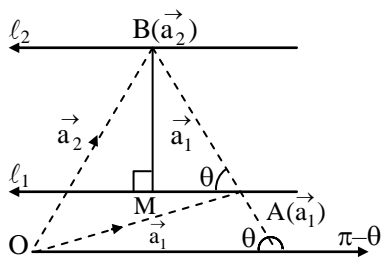
The lines are intersecting, if shortest distance = 0

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

(ii) **Distance between parallel lines :** Let

$l_1$  and  $l_2$  are two parallel lines whose equations are

$$\vec{r} = \vec{a}_1 + \lambda \vec{b} \quad \text{and} \quad \vec{r} = \vec{a}_2 + \mu \vec{b} \quad \text{respectively}$$



Clearly,  $l_1$  and  $l_2$  Pass through the points A and B with position vectors  $\vec{a}_1$  and  $\vec{a}_2$  respectively and both are parallel to the vector  $\vec{b}$ , where BM is the shortest distance between  $l_1$  and  $l_2$

shortest distance between parallel lines :

$$\vec{r} = \vec{a}_1 + \lambda \vec{b} \text{ and } \vec{r} = \vec{a}_2 + \mu \vec{b} \text{ is :}$$

$$d = \frac{|(\vec{a}_2 - \vec{a}_1) \times \vec{b}|}{|\vec{b}|}$$

## 8. Plane

A plane is a surface such that if any two points are taken on it, the line segment joining them lies completely on the surface. In other words, every point on the line segment joining any two points lies on the plane.

**Theorem :** Every first degree equation in x, y and z represents a plane i.e.  $ax+by+cz+d=0$  is the general equation of a plane.

### Equation of a plane passing through a given point

The general equation of a plane passing through a point  $(x_1, y_1, z_1)$  is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0, \text{ where } a, b \text{ and } c \text{ are constants.}$$

### Intercept form of a plane :

The equation of a plane intercepting lengths a, b and c with x- axis , y-axis and z-axis respectively is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

**Cartesian Form :** If  $\ell, m, n$  are direction cosines of the normal to a given plane which is at a distance p from the origin, then the equation of the plane is  $\ell x + my + nz = p$ .

## 9. Normal to a plane

A line perpendicular to a plane is called a normal to the plane. Clearly, every line lying in a plane is perpendicular to the normal to the plane.

**For example :** The direction ratios of a vector normal to the plane  $3x + 2y + 5z - 6 = 0$  are 3, 2, 5 and hence a vector normal to the plane is  $3\hat{i} + 2\hat{j} + 5\hat{k}$ .

### (i) Vector equation of plane passing through a point and normal to a given vector

The vector equation of a plane passing through a point having position vector  $\vec{a}$  is  $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$

### Reduction to cartesian form :

If  $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$  ,  $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$  and  $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$

Then ;  $(\vec{r} - \vec{a}) = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$

Then (i) can be written as

$$\{(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}\} \cdot \{a\hat{i} + b\hat{j} + c\hat{k}\} = 0$$

$$\{(a\hat{i} + b\hat{j} + c\hat{k})\} = 0$$

$$\Rightarrow a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

Thus, the coefficient of x, y, z in the cartesian equation of a plane are the direction ratios of normal to the plane.

### (ii) Equation of plane in normal form

#### vector form

The vector equation of a plane normal to unit vector  $\hat{n}$  and at a distance d from the origin

$$\text{is } \vec{r} \cdot \hat{n} = d$$

#### Cartesian form

If  $\ell, m, n$ , be the direction cosines of the normal to a given plane and p be the length of perpendicular from origin to the plane, then the equation of the plane is  $\ell x + my + nz = p$ .

## 10. Angle between two planes



- (i) **Vector form** - The angle between the two planes is defined as the angle between normals.

Let  $\theta$  be the angle between planes;

$$\vec{r} \cdot \vec{n}_1 = d_1 \text{ and } \vec{r} \cdot \vec{n}_2 = d_2 \text{ is given by}$$

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|}$$

- (ii) **Cartesian form** - The angle  $\theta$  between the planes  $a_1 x + b_1 y + c_1 z + d_1 = 0$  and

$a_2 x + b_2 y + c_2 z + d_2 = 0$  is given by

$$\cos \theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

## 11. Intersection of two planes

The equation of a plane passing through the intersection of  $a_1 x + b_1 y + c_1 z + d_1 = 0$  and  $a_2 x + b_2 y + c_2 z + d_2 = 0$  is  $(a_1 x + b_1 y + c_1 z + d_1) + \lambda (a_2 x + b_2 y + c_2 z + d_2) = 0$ , where  $\lambda$  is a constant.

## ➤ Distance of a point from a plane

- (i) **Vector form** : - The length of the perpendicular from a point having position vector  $\vec{a}$  to, the

$$\text{plane } \vec{r} \cdot \vec{n} = d \text{ is given by } P = \frac{|\vec{a} \cdot \vec{n} - d|}{|\vec{n}|}$$

- (ii) **Cartesian Form** : The length of the perpendicular from a point  $P(x_1, y_1, z_1)$  to the plane  $ax + by + cz + d = 0$  is given by

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

## ➤ Distance between the parallel planes

- (i) **Vector form** : The distance between two parallel plane  $\vec{r} \cdot \vec{n} = d_1$

and  $\vec{r} \cdot \vec{n} = d_2$  is given by

$$d = \frac{|d_1 - d_2|}{|\vec{n}|}$$

- (ii) **Cartesian form**

The distance between two parallel planes

$$ax + by + cz + d_1 = 0 \text{ and}$$

$ax + by + cz + d_2 = 0$  is given by

$$d = \left| \frac{(d_2 - d_1)}{\sqrt{a^2 + b^2 + c^2}} \right|$$

## ➤ Equation of planes bisecting the angles between two given planes

**Cartesian Form** :

The equation of the planes bisecting the angles between the planes  $a_1 x + b_1 y + c_1 z + d_1 = 0$  and  $a_2 x + b_2 y + c_2 z + d_2 = 0$  are

$$\frac{(a_1 x + b_1 y + c_1 z + d_1)}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{(a_2 x + b_2 y + c_2 z + d_2)}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

## 12. Lines and plane

### 12.1 Angle between a line and a plane

The angle between a line and a plane is the complement of the angle between the line and the normal to the plane

If  $\alpha, \beta, \gamma$  be the direction ratios of the line and  $ax + by + cz + d = 0$  be the equation of plane and  $\theta$  be the angle between the line and the plane.

$$\Rightarrow \cos(90^\circ - \theta) = \frac{a\alpha + b\beta + c\gamma}{\sqrt{a^2 + b^2 + c^2} \sqrt{\alpha^2 + \beta^2 + \gamma^2}}$$

$$\text{or } \sin \theta = \frac{a\alpha + b\beta + c\gamma}{\sqrt{a^2 + b^2 + c^2} \sqrt{\alpha^2 + \beta^2 + \gamma^2}}$$

**Vector form** : If  $\theta$  is the angle between the line;

$$\vec{r} = \vec{a} + \lambda \vec{b} \text{ and plane } \vec{r} \cdot \vec{n} = d$$

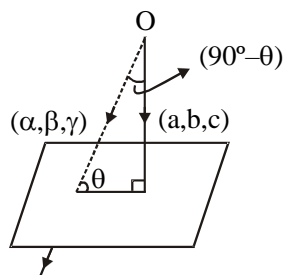
$$\Rightarrow \sin \theta = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}| |\vec{n}|}$$

### 12.2 Condition for a line to be parallel to a plane

Let line  $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$  be parallel to plane  $ax + by + cz + d = 0$  iff;

$$\theta = 0 \text{ or } \pi \text{ or } \sin \theta = 0$$

or  $al + bm + cn = 0$



### 12.3 Condition for a line to lie in the plane

Condition for  $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$  to lie in the plane

$ax + by + cz + d = 0$  are

$al + bm + cn = 0$  and

$ax_1 + by_1 + cz_1 + d = 0$

**Note :** A line will be in a plane iff.

- (i) the normal to the plane is perpendicular to the line
- (ii) a point on the line lies in the plane.

### 13. Condition of coplanarity of two lines & equation of the plane containing them

**Cartesian form :**

If the line  $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$  and

$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$  are coplanar, then

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

and the equation of the plane containing them is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

or  $\begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

### 14.1 Volume of a Tetrahedron :

Volume of a tetrahedron with vertices A  $(x_1, y_1, z_1)$ , B  $(x_2, y_2, z_2)$  C  $(x_3, y_3, z_3)$  and D  $(x_4, y_4, z_4)$  is given by

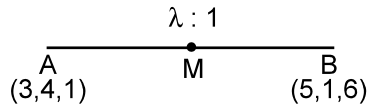
$$V = \frac{1}{6} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

### 14. Miscellaneous point

## SOLVED EXAMPLE

**Ex.1** Find the co-ordinates of the point where the line joining A (3, 4, 1) and B (5, 1, 6) crosses the xy-plane.

**Sol.** Let the line joining A and B crosses the xy-plane at the point M and let M divides AB in the ratio  $\lambda : 1$  internally.



Co-ordinates of M are

$$\left[ \frac{5\lambda + 3}{\lambda + 1}, \frac{\lambda + 4}{\lambda + 1}, \frac{6\lambda + 1}{\lambda + 1} \right]$$

Since the point M lies on the xy-plane

Its z-co-ordinate is zero.

$$\therefore \frac{6\lambda + 1}{\lambda + 1} = 0 \Rightarrow 6\lambda + 1 = 0$$

$$\text{or } \lambda = -\frac{1}{6}$$

The ratio is  $\lambda : 1 = \frac{1}{6} : 1$  i.e. 1 : 6 externally.

Hence the co-ordinates of M are

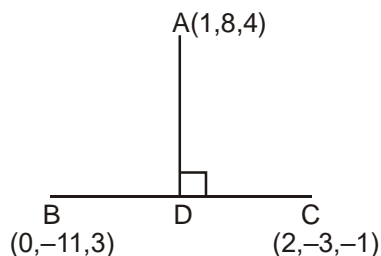
$$\text{i.e. } \left[ \frac{5(-1/6) + 3}{(-1/6) + 1}, \frac{(-1/6) + 4}{(-1/6) + 1}, 0 \right],$$

$$\left( \frac{13}{5}, \frac{23}{5}, 0 \right). \quad \text{Ans.}$$

**Ex.2** A (1, 8, 4), B (0, -11, 3), C (2, -3, -1) are three points and D is the foot of the perpendicular from A on BC. Find the co-ordinate of D.

**Sol.** Let D be the foot of the  $\perp$  from A on BC, If P divides BC in the ratio  $k : 1$ , then co-ordinates

of D are  $\left( \frac{2k + 0}{k + 1}, \frac{-3k - 11}{k + 1}, \frac{-k + 3}{k + 1} \right)$



D.R.'s of AD are

$$\frac{2k}{k+1} - 1, \frac{-3k-11}{k+1} - 8, \frac{-k+3}{k+1} - 4.$$

$$\Rightarrow \frac{k-1}{k+1}, \frac{-11k-19}{k+1}, \frac{-5k-1}{k+1}$$

D.R.'s of BC are  $2-0, -3+11, -1-3,$

i.e., 2, 8, -4

Since  $AD \perp BC$

$$\therefore \frac{k-1}{k+1} \times 2 + \frac{-11k-19}{k+1} \times 8 + \frac{-5k-1}{k+1} \times (-4) = 0$$

$$\Rightarrow \frac{2k-2-88k-152+20k+4}{k+1} = 0$$

$$\Rightarrow -66k - 150 = 0 \Rightarrow 66k = -150$$

$$\therefore k = -\frac{25}{11}$$

Hence the co-ordinates of D are

$$\left( \frac{25}{7}, \frac{23}{7}, -\frac{4}{7} \right) \quad \text{Ans.}$$

**Ex.3** The distance of centroid from x-axis of the triangle formed by the points (2, -4, 3), (3, -1, -2) and (-2, 5, 8) is-

(A) 1 (B) 0

(C) 3 (D)  $\sqrt{10}$

**Sol.** The centroid of the triangle is -

$$\left( \frac{2+3-2}{3}, \frac{-4-1+5}{3}, \frac{3-2+8}{3} \right) \equiv (1, 0, 3)$$

$$\text{Its distance from x-axis} = \sqrt{0^2 + 3^2} = 3$$

**Ans.**

**Ex.4** A line OP makes with the x-axis an angle of measure  $120^\circ$  and with y-axis an angle of measure  $60^\circ$ . Find the angle made by the line with the z-axis.

**Sol.**  $\alpha = 120^\circ$  and  $\beta = 60^\circ$

$$\therefore \cos \alpha = \cos 120^\circ = -\frac{1}{2}$$

$$\text{and } \cos \beta = \cos 60^\circ = \frac{1}{2}$$

$$\text{But } \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

$$\therefore \left( \frac{-1}{2} \right)^2 + \left( \frac{1}{2} \right)^2 + \cos^2 \gamma = 1$$

$$\cos^2 \gamma = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$$

$$\therefore \cos \gamma = \pm \frac{1}{\sqrt{2}}$$

$$\therefore \gamma = 45^\circ \text{ or } 135^\circ. \quad \text{Ans.}$$

**Ex.5** The locus of a point, which moves in such a way that its distance from the origin is thrice the distance from xy-plane is -

(A)  $x^2 - 8y^2 - 8z^2 = 0$

(B)  $x^2 - 8y^2 + z^2 = 0$

(C)  $-8x^2 + y^2 + z^2 = 0$

(D)  $x^2 + y^2 - 8z^2 = 0$

**Sol.** Let the point be P (x, y, z), then its distance from origin is

$$OP = \sqrt{x^2 + y^2 + z^2}$$

The distance of P from xy-plane is z.

According to question

$$\sqrt{x^2 + y^2 + z^2} = 3z$$

Thus the required locus is

$$x^2 + y^2 - 8z^2 = 0. \quad \text{Ans.}$$

**Ex.6** Find the measure of the angle between the lines whose direction ratios are 1, -2, 7 and 3, -2, -1.

**Sol.**  $\sqrt{1^2 + (-2)^2 + 7^2} = \sqrt{54}$

$$\sqrt{3^2 + (-2)^2 + (-1)^2} = \sqrt{14}$$

$\therefore$  The actual direction cosines of the lines are

$$\frac{1}{\sqrt{54}}, \frac{-2}{\sqrt{54}}, \frac{7}{\sqrt{54}} \text{ and } \frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}$$

If  $\theta$  is the angle between the lines, then

$$\cos \theta =$$

$$\left( \frac{1}{\sqrt{54}} \right) \left( \frac{3}{\sqrt{14}} \right) + \left( \frac{-2}{\sqrt{54}} \right) \left( \frac{-2}{\sqrt{14}} \right) + \left( \frac{7}{\sqrt{54}} \right) \left( \frac{-1}{\sqrt{14}} \right)$$

$$= \frac{3 + 4 - 7}{\sqrt{54} \cdot \sqrt{14}} = 0 \Rightarrow \theta = 90^\circ. \quad \text{Ans.}$$

**Ex.7** Find the projection of the line segment joining the points (-1, 0, 3) and (2, 5, 1) on the line whose direction ratios are 6, 2, 3.

**Sol.** The direction cosines  $\ell$ ,  $m$ ,  $n$  of the line are given by

$$\frac{\ell}{6} = \frac{m}{2} = \frac{n}{3} = \frac{\sqrt{\ell^2 + m^2 + n^2}}{\sqrt{6^2 + 2^2 + 3^2}} = \frac{1}{\sqrt{49}} = \frac{1}{7}$$

$$\therefore \ell = \frac{6}{7}, m = \frac{2}{7}, n = \frac{3}{7}$$

The required projection is given by

$$= \ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

$$= \frac{6}{7}[2 - (-1)] + \frac{2}{7}(5 - 0) + \frac{3}{7}(1 - 3)$$

$$= \frac{6}{7} \times 3 + \frac{2}{7} \times 5 + \frac{3}{7} \times -2$$

$$= \frac{18}{7} + \frac{10}{7} - \frac{6}{7} = \frac{18 + 10 - 6}{7} = \frac{22}{7}. \quad \text{Ans.}$$

**Ex.8** The foot of the perpendicular drawn from the point P (1, 0, 3) to the join of points A (4, 7, 1) and B (3, 5, 3) is -

(1) (5, 7, 1)                      (2)  $\left( \frac{5}{3}, \frac{7}{3}, \frac{17}{3} \right)$

(3)  $\left( \frac{2}{3}, \frac{5}{3}, \frac{7}{3} \right)$                       (4)  $\left( \frac{5}{3}, \frac{2}{3}, \frac{7}{3} \right)$

**Sol.** Drawn  $PL \perp AB$ . Suppose L divides AB in the ratio  $k : 1$ , then coordinates of L are

$$\left( \frac{3k+4}{k+1}, \frac{5k+7}{k+1}, \frac{3k+1}{k+1} \right)$$

Direction ratios of AB are 1, 2, -2 and

Direction ratios of, PL are

$$\frac{2k+3}{k+1}, \frac{5k+7}{k+1}, \frac{-2}{k+1}$$

Since  $PL \perp AB$ , We have

$$\left( \frac{2k+3}{k+1} \right) \times (1) + \left( \frac{5k+7}{k+1} \right) \times (2) + \left( \frac{-2}{k+1} \right) \times (-2) = 0$$

$$\Rightarrow k = -7/4$$

$$\therefore L \text{ is } \left( \frac{5}{3}, \frac{7}{3}, \frac{17}{3} \right) \quad \text{Ans.}$$

**Ex.9** If two points are P (7, -5, 11) and Q (-2, 8, 13), then projection of PQ on a straight line with direction cosines

$$\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \text{ is-}$$



- (A) 7 (B) 8  
(C) 9 (D) none of these

**Sol.** The projection of line joining the points P (7, -5, 11) and Q (-2, 8, 13) on a line with direction cosines  $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$  is

$$= (-2-7) \frac{1}{3} + (8+5) \frac{2}{3} + (13-11) \frac{2}{3}$$

$$= -3 + \frac{26}{3} + \frac{4}{3} = 7 \quad \text{Ans.}$$

**Ex.10** Find the ratio in which the plane  $2x + 3y + 5z = 1$  divides the line joining the points (1, 0, -3) and (1, -5, 7).

**Sol.** Let the required ratio be k : 1  
The co-ordinates of the point which divide the join of (1, 0, -3) and (1, -5, 7) in the ratio k : 1 are

$$\left( \frac{k+1}{k+1}, \frac{-5k}{k+1}, \frac{7k-3}{k+1} \right)$$

Since this point lies on  $2x + 3y + 5z = 1$ ,

$$\therefore 2 \left( \frac{k+1}{k+1} \right) + 3 \left( \frac{-5k}{k+1} \right) + 5 \left( \frac{7k-3}{k+1} \right) = 1$$

$$\Rightarrow 2k + 2 - 15k + 35k - 15 = k + 1$$

$$\Rightarrow 22k - 13 = k + 1$$

$$\Rightarrow 21k = 14; \quad k = \frac{14}{21} = \frac{2}{3}$$

Hence the required ratio =

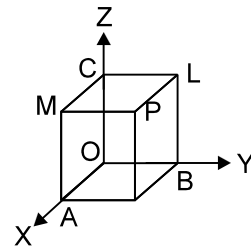
$$k : 1 = \frac{2}{3} : 1 = 2 : 3. \quad \text{Ans.}$$

**Ex.11** If a line makes angles  $\alpha, \beta, \gamma, \delta$  with four diagonals of a cube, then  $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta$  equals -

- (A) 3 (B) 4  
(C) 4/3 (D) 3/4

**Sol.** Let OA, OB, OC be coterminous edges of a cube and OA = OB = OC = a, then co-ordinates of its vertices are O (0, 0, 0), A (a, 0, 0), B (0, a, 0), C (0, 0, c), L (0, a, a), M (a, 0, a), N (a, a, 0) and P (a, a, a)

Direction ratio of diagonal AL, BM, CN and OP are



$$\left( -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right), \left( \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right),$$

$$\left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right), \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

Let  $\ell, m, n$  be the direction cosines of the given line, then

$$\cos \alpha = \ell \left( -\frac{1}{\sqrt{3}} \right) + m \left( \frac{1}{\sqrt{3}} \right) + n \left( \frac{1}{\sqrt{3}} \right)$$

$$= \frac{-\ell + m + n}{\sqrt{3}}$$

$$\text{Similarly } \cos \beta = \frac{\ell - m + n}{\sqrt{3}},$$

$$\cos \gamma = \frac{\ell + m - n}{\sqrt{3}} \quad \text{and} \quad \cos \delta = \frac{\ell + m + n}{\sqrt{3}}$$

$$\therefore \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = \frac{4}{3} \quad \text{Ans.}$$

**Ex.12** The cosine of angle between any two diagonal of a cube is -

- (A) 1/3 (B) 1/2  
(C) 2/3 (D)  $1/\sqrt{3}$

**Sol.** From above example, we have direction ratios of diagonals AL, BM, CN and OP as

$$\left( -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right), \left( \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right),$$

$$\left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right), \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

If  $\theta_1$  be the angle between OP and AL, then

$\cos \theta_1$

$$= \left\{ \left( \frac{1}{\sqrt{3}} \right) \left( -\frac{1}{\sqrt{3}} \right) + \left( \frac{1}{\sqrt{3}} \right) \left( \frac{1}{\sqrt{3}} \right) + \left( \frac{1}{\sqrt{3}} \right) \left( \frac{1}{\sqrt{3}} \right) \right\}$$

$$= \frac{1}{3}$$

$$\Rightarrow \theta_1 = \cos^{-1} \left( \frac{1}{3} \right)$$

Similarly, the angle between each of the other pairs of diagonals is  $\cos^{-1} \left( \frac{1}{3} \right)$  **Ans.**

**Ex.13** The distance of the point  $(1, -2, 3)$  from the plane  $x - y + z = 5$  measured parallel to the

line  $\frac{x}{2} = \frac{y}{3} = \frac{z-1}{-6}$  is-

- (A) 1 (B) 2  
(C) 4 (D) None of these

**Sol.** Equation of the line through  $(1, -2, 3)$

parallel to the line  $\frac{x}{2} = \frac{y}{3} = \frac{z-1}{-6}$

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} = r \text{ (say)} \quad \dots(1)$$

Then any point on (1) is

$$(2r + 1, 3r - 2, -6r + 3).$$

If this point lies on the plane  $x - y + z = 5$ , then  $(2r + 1) - (3r - 2) + (-6r + 3) = 5$

$$\Rightarrow -7r + 6 = 5, \text{ i.e., } r = \frac{1}{7}$$

Hence the point is  $\left( \frac{9}{7}, -\frac{11}{7}, \frac{15}{7} \right)$ .

Distance between  $(1, -2, 3)$  and

$$\left( \frac{9}{7}, -\frac{11}{7}, \frac{15}{7} \right)$$

$$= \sqrt{\left( \frac{4}{49} + \frac{9}{49} + \frac{36}{49} \right)} = \sqrt{\left( \frac{49}{49} \right)} = 1$$

**Ans.**

**Ex.14** The equation of the plane containing the line

$\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$  and the point  $(0, 7, -7)$

is-

- (A)  $x + y + z = 2$  (B)  $x + y + z = 3$   
(C)  $x + y + z = 0$  (D) None of these

**Sol** Any plane containing  $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$

is

$$a(x + 1) + b(y - 3) + c(z + 2) = 0, \dots(1)$$

$$\text{where } -3a + 2b + c = 0 \quad \dots(2)$$

If the plane through  $(0, 7, -7)$ , then

$$a + 4b - 5c = 0 \quad \dots(3)$$

From (2) and (3),

$$\frac{a}{-10-4} = \frac{b}{1-15} = \frac{c}{-12-2}$$

$$\text{i.e. } \frac{a}{1} = \frac{b}{1} = \frac{c}{1}$$

Hence the plane (1) becomes

$$(x + 1) + (y - 3) + (z + 2) = 0,$$

$$\text{i.e. } x + y + z = 0$$

**Ans.**