DETERMINANTS

(KEY CONCEPTS + SOLVED EXAMPLES)

DETERMINANTS

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KEY CONCEPTS

1. Definition

We know how to solve an equation of the type $ax + b = 0$. This has the solution $x = -\frac{b}{a}$ provided $a \neq 0$. Now consider the two equation of this type

$$ax + b = 0 \qquad ...(i)$$

$$cx + d = 0 \qquad ...(ii)$$

If these two equations are satisfied by the same value of x, they are said to be consistent. i.e.

If $-\frac{b}{a} = -\frac{d}{c}$

if $ad = bc$

if $ad - bc = 0$

The expression $ad - bc$ is called the eliminant for the equations (i) and (ii). If we write the coefficient of the equations in the following way $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, then such an arrangement is called a determinant of order 2 and its value is defined to be $ad - bc$ which is our eliminant,

Thus

A determinant is a special kind of symbol used into determine certain properties of systems of equations and functions. Many complicated expressions can be easily handled if they are expressed as determinants

or

An expression expressed in equal number of rows and column and put between two vertical lines is named as determinant of that expression e.g. $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ are the determinants of second and third order respectively.

2. Expansion of Determinant

Unlike a matrix, determinant is not just a table of numerical data but (quite differently) a short hand way of writing an algebraic expression, whose value can be computed when the values of terms or elements are known.

(i) The 4 numbers a_1, b_1, a_2, b_2 arranged as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is a determinant of second order. These numbers are called the elements of the determinant. The value of the determinant is defined as

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

The expanded form of determinant has 2! terms.

(ii) The 9 numbers a_r, b_r, c_r $(r = 1, 2, 3)$ arranged as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is determinant of third order.

Take any row (or column) ; the value of the determinant is the sum of products of the elements of the row (or column) and the corresponding determinant obtained by omitting the row and the column of the element with a proper sign, given by the rule $(-1)^{r+s}$, where r and s are the number of rows and the number of column respectively of the element of the row (or the column) chosen

Thus

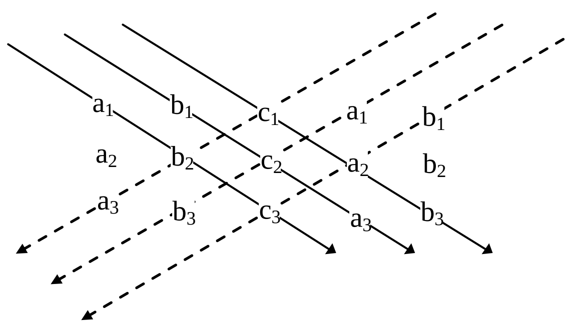
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$$

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

The diagonal through the left-hand top corner which contains the element a_1, b_2, c_3 is called the leading diagonal or principal diagonal and the terms are called the leading terms. The expanded form of determinant has 3! terms

Short cut

To find the value of third order determinant, following method is also useful



Taking product of R.H.S. diagonal element positive and L.H.S. diagonal elements negative and adding them. We get the value of Determinant as =

$a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3 - a_1c_2b_3 - b_1a_2c_3$

3. Minor and Cofactor

Minor

The Determinant that is left by cancelling the row and column intersecting at a particular element is called the minor of that element.

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ then Minor of a_{11} is

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \text{ Similarly } M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Using this concept the value of Determinant can be

$$\Delta = a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13}$$

or $\Delta = -a_{21} M_{21} + a_{22} M_{22} - a_{23} M_{23}$

or $\Delta = a_{31} M_{31} - a_{32} M_{32} + a_{33} M_{33}$

Cofactor

The cofactor of an element a_{ij} is denoted by F_{ij} and is equal to $(-1)^{i+j} M_{ij}$ where M is a minor of element a_{ij}

if $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

then $F_{11} = (-1)^{1+1} M_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$$F_{12} = (-1)^{1+2} M_{12} = -M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Note :

(i) The sum of products of the element of any row with their corresponding cofactor is equal to the value of determinant i.e.

$$\Delta = a_{11}F_{11} + a_{12}F_{12} + a_{13}F_{13}$$

(ii) The sum of the product of element of any row with corresponding cofactor of another row is equal to zero i.e.

$$a_{11}F_{21} + a_{12}F_{22} + a_{13}F_{23} = 0$$

(iii) If order of a determinant (Δ) is 'n' then the value of the determinant formed by replacing every element by its cofactor is Δ^{n-1}

4. Properties of Determinant

P-1 The value of Determinant remains unchanged, if the rows and the column are interchanged. This is always denoted by ' and is also called transpose

Note :

Since the Determinant remains unchanged when rows and columns are interchanged, it is obvious that any theorem which is true for 'rows' must also be true for 'Columns'

P-2 If any two rows (or columns) of a determinant be interchanged, the determinant is unaltered in numerical Value, but is changed in sign only,

P-3 If a Determinant has two rows (or columns) identical, then its value is zero.

P-4 If all the elements of any row (or column) be multiplied by the same number, then the value of Determinant is multiplied by that number.

P-5 If each element of any row (or column) can be expressed as a sum of two terms, then the determinant can be expressed as the sum of the Determinants

P-6 The value of a Determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column)

Note :

It should be noted that while applying P-6 at least one row (or column) must remain unchanged

P-7 If $\Delta = f(x)$ and $f(a) = 0$ then $(x-a)$ is a factor of Δ

5. Multiplication of Two Determinants

Multiplication of two second order determinants is defined as follows

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \ell_1 & m_1 \\ \ell_2 & m_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\ell_1 + b_1\ell_2 & a_1m_1 + b_1m_2 \\ a_2\ell_1 + b_2\ell_2 & a_2m_1 + b_2m_2 \end{vmatrix}$$

Multiplication of two third order determinants is defined as follows

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\ell_1 + b_1\ell_2 + c_1\ell_3 & a_1m_1 + b_1m_2 + c_1m_3 & a_1n_1 + b_1n_2 + c_1n_3 \\ a_2\ell_1 + b_2\ell_2 + c_2\ell_3 & a_2m_1 + b_2m_2 + c_2m_3 & a_2n_1 + b_2n_2 + c_2n_3 \\ a_3\ell_1 + b_3\ell_2 + c_3\ell_3 & a_3m_1 + b_3m_2 + c_3m_3 & a_3n_1 + b_3n_2 + c_3n_3 \end{vmatrix}$$

Note :

In above case the order of Determinant is same, if the order is different then for their multiplication first of all they should be expressed in the same order.

6. Symmetric & Skew symmetric Determinant

Symmetric determinant

A determinant is called symmetric Determinant if for its every element.

$a_{ij} = a_{ji} \ \forall \ i, j$

Skew Symmetric determinant

A determinant is called skew Symmetric determinant if for its every element

$a_{ij} = -a_{ji} \ \forall \ i, j$

Note :

(i) Every diagonal element of a skew symmetric determinant is always zero

(ii) The value of a skew symmetric determinant of even order is always a perfect square and that of odd order is always zero.

7. Applications of Determinant

Crammer's Rule

Consider three linear simultaneous equation in 'x', 'y', 'z'

$a_1x + b_1y + c_1z = d_1$ ...(i)

$a_2x + b_2y + c_2z = d_2$ ...(ii)

$a_3x + b_3y + c_3z = d_3$ ...(iii)

and

$$\text{if } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \quad \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

then using Crammer's rule of determinant we get

$$\frac{x}{\Delta_1} = \frac{y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{1}{\Delta}$$

i.e. $x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$, $z = \frac{\Delta_3}{\Delta}$

Case-I If $\Delta \neq 0$

Then $x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$, $z = \frac{\Delta_3}{\Delta}$

$\therefore$ The system is consistent and has unique solutions

Case-II If $\Delta = 0$ and

(i) If at least one of Δ_1, Δ_2, Δ_3 is not zero then the system of equations is inconsistent i.e. has no solution

(ii) If $d_1 = d_2 = d_3 = 0$ or $\Delta_1, \Delta_2, \Delta_3$ are all zero then the system of equations has infinitely many solutions.

The above can be shown diagrammatically as follows

Find Δ

$\Delta \neq 0$
Unique solution
$x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$, $z = \frac{\Delta_3}{\Delta}$
Equations are consistent

$\Delta = 0$

$\Delta_1, \Delta_2, \Delta_3$ are not all zero

No solution
Equations are inconsistent

$\Delta_1, \Delta_2, \Delta_3$ are all zero

Infinitely many solution
Equations are consistent

SOLVED EXAMPLES

Ex.1 The value of the determinant

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} \text{ is -}$$

(A) $2(a + b + c)$ (B) $2(a + b + c)^2$

(C) $2(a + b + c)^3$ (D) $(2a + 2b + 2c)^3$

Sol. Applying $C_1 + C_2 + C_3$, we get

$$\text{Det.} = 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & 0 & c+a+b \end{vmatrix}$$

[by $R_2 - R_1$, $R_3 - R_1$]

$= 2(a + b + c)^3$ **Ans.[C]**

Ex.2 $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix} = 0$, then a, b, c are in -

(A) A.P. (B) G.P.

(C) H.P. (D) None of these

Sol. By the operation $C_3 - (\alpha C_1 + C_2)$, we get

$$\begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha+b & b\alpha+c & -(a\alpha^2+2b\alpha+c) \end{vmatrix} = 0$$

$\Rightarrow -(a\alpha^2 + 2b\alpha + c)(ac - b^2) = 0$

$\Rightarrow b^2 = ac \Rightarrow$ a, b, c are in G.P.

Ans.[B]

Ex.3 $\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$ is equal to -

(A) $a^2b^2c^2$ (B) $2a^2b^2c^2$

(C) $4a^2b^2c^2$ (D) None of these

Sol. Applying $R_1 - (R_2 + R_3)$, we get

$$\text{Det.} = \begin{vmatrix} 0 & -2c^2 & -2b^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & -c^2 & -b^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & -c^2 & -b^2 \\ b^2 & a^2 & 0 \\ c^2 & 0 & a^2 \end{vmatrix}$$

(by $R_2 + R_1$, $R_3 + R_1$)

$= 2(a^2b^2c^2 + a^2b^2c^2) = 4a^2b^2c^2$

Ans.[C]

Ex.4 If $\begin{vmatrix} a & 5x & p \\ b & 10y & 5 \\ c & 15z & 15 \end{vmatrix} = 125$, then $\begin{vmatrix} 3a & 3b & c \\ x & 2y & z \\ p & 5 & 5 \end{vmatrix}$ is equal to -

(A) 25 (B) 50

(C) 75 (D) 100

Sol. $\begin{vmatrix} 3a & 3b & c \\ x & 2y & z \\ p & 5 & 5 \end{vmatrix} = \begin{vmatrix} 3a & x & p \\ 3b & 2y & 5 \\ c & z & 5 \end{vmatrix}$

(changing rows into columns)

$$= \frac{1}{3}\begin{vmatrix} 3a & x & p \\ 3b & 2y & 5 \\ 3c & 3z & 15 \end{vmatrix}$$

$$= \frac{3}{3} \times \frac{1}{5}\begin{vmatrix} a & 5x & p \\ b & 10y & 5 \\ c & 15z & 15 \end{vmatrix} = \frac{1}{5}(125) = 25.$$

Ans.[A]

Ex.5 $\Delta = \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}$, then the value of

$$\Delta' = \begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2+\lambda^2 & bc+a\lambda \\ ac+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix} \text{ is -}$$

(A) 3Δ (B) Δ^2

(C) Δ^3 (D) None of these

Sol. Here the cofactors of λ, c, $-$ b,.......in Δ are $a^2 + \lambda^2$, $ab + c\lambda$, $ca - b\lambda$,.....respectively.

Therefore the value of Δ' is Δ^2. **Ans.[B]**

Ex.6 If $\begin{vmatrix} 3^2+k & 4^2 & 3^2+3+k \\ 4^2+k & 5^2 & 4^2+4+k \\ 5^2+k & 6^2 & 5^2+5+k \end{vmatrix} = 0$, then the value of k is –

(A) 2 (B) 1 (C) –1 (D) 0

Sol. Breaking the given determinant into two determinants, we get

$$\begin{vmatrix} 3^2+k & 4^2 & 3^2+k \\ 4^2+k & 5^2 & 4^2+k \\ 5^2+k & 6^2 & 5^2+k \end{vmatrix} + \begin{vmatrix} 3^2+k & 4^2 & 3 \\ 4^2+k & 5^2 & 4 \\ 5^2+k & 6^2 & 5 \end{vmatrix} = 0$$

$$\Rightarrow 0 + \begin{vmatrix} 9+k & 16 & 3 \\ 7 & 9 & 1 \\ 9 & 11 & 1 \end{vmatrix} = 0$$

[Applying $R_3 - R_2$ and $R_2 - R_1$ in second det.]

$$\Rightarrow \begin{vmatrix} 9+k & 16 & 3 \\ 7 & 9 & 1 \\ 2 & 2 & 0 \end{vmatrix} = 0 \text{ [Applying } R_3 - R_2]$$

$$\Rightarrow \begin{vmatrix} 9+k & 7-k & 3 \\ 7 & 2 & 1 \\ 2 & 0 & 0 \end{vmatrix} = 0 \text{ [Applying } C_2 - C_1]$$

$\Rightarrow$ $2(7 - k - 6) = 0$

$\Rightarrow$ $k = 1$. **Ans.[B]**

Ex.7 The determinant $\begin{vmatrix} 0 & (a-b)^2 & (a-c)^2 \\ (b-a)^2 & 0 & (b-c)^2 \\ (c-a)^2 & (c-b)^2 & 0 \end{vmatrix}$ is equal to -

(A) $(a-b)^2 (b-c)^2 (c-a)^2$

(B) 0

(C) $2(a-b)^2 (b-c)^2 (c-a)^2$

(D) None of these

Sol. Expanding the det., we get

$\Delta = -(b-a)^2 [0 - (a-c)^2 (c-b)^2] + (c-a)^2 [(a-b)^2 (b-c)^2 - 0]$

$= 2(a-b)^2 (b-c)^2 (c-a)^2$. **Ans.[C]**

Ex.8 If $0 < \theta < \pi/2$ and

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0 \text{ then}$$

θ is equal to

(A) $\pi/24, 5\pi/24$ (B) $5\pi/24, 7\pi/24$

(C) $7\pi/24, 11\pi/24$ (D) None of these

Sol. Applying $R_2 - R_1$ and $R_3 - R_1$, We get

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow 2 + 4\sin 4\theta = 0$

$\Rightarrow \sin 4\theta = -1/2$

$\Rightarrow 4\theta = n\pi + (-1)^n (-\pi/6)$

$\Rightarrow \theta = n\pi/4 + (-1)^n(-\pi/24)$

$\therefore \theta = 7\pi/24, 11\pi/24$. **Ans.[C]**

Ex.9 $\begin{vmatrix} {}^xC_1 & {}^xC_2 & {}^xC_3 \\ {}^yC_1 & {}^yC_2 & {}^yC_3 \\ {}^zC_1 & {}^zC_2 & {}^zC_3 \end{vmatrix}$ is equal to -

(A) $xyz (x-y)(y-z)(z-x)$

(B) $\frac{xyz}{6} (x-y)(y-z)(z-x)$

(C) $\frac{xyz}{12} (x-y)(y-z)(z-x)$

(D) None of these

Sol. Det. $= \begin{vmatrix} x & \frac{x(x-1)}{2} & \frac{x(x-1)(x-2)}{6} \\ y & \frac{y(y-1)}{2} & \frac{y(y-1)(y-2)}{6} \\ z & \frac{z(z-1)}{2} & \frac{z(z-1)(z-2)}{6} \end{vmatrix}$

$$= \frac{xyz}{12} \begin{vmatrix} 1 & x-1 & (x-1)(x-2) \\ 1 & y-1 & (y-1)(y-2) \\ 1 & z-1 & (z-1)(z-2) \end{vmatrix} = \frac{xyz}{12} \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

(by $C_2 + C_1$, $C_3 + C_1 + 3C_2$)

$= \frac{xyz}{12} (x-y)(y-z)(z-x)$ **Ans.[C]**

Ex.10 If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$ then

(A) $\Delta_1 = 3\Delta_2^2$ (B) $\frac{d}{dx}(\Delta_1) = 3\Delta_2^2$

(C) $\frac{d}{dx}(\Delta_1) = 3\Delta_2$ (D) None of these

Sol. Since we know that if $\Delta = f(x) = \begin{vmatrix} R_1 \\ R_2 \\ R_3 \end{vmatrix}$, then

$$\frac{d}{dx}(\Delta) = \begin{vmatrix} \frac{d}{dx}(R_1) \\ R_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ \frac{d}{dx}(R_2) \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2 \\ \frac{d}{dx}(R_3) \end{vmatrix}$$

$$\therefore \frac{d}{dx}(\Delta_1) = \begin{vmatrix} \frac{d}{dx}(x) & \frac{d}{dx}(b) & \frac{d}{dx}(b) \\ a & x & b \\ a & a & x \end{vmatrix}$$

$$+ \begin{vmatrix} x & b & b \\ \frac{d}{dx}(a) & \frac{d}{dx}(x) & \frac{d}{dx}(b) \\ a & a & x \end{vmatrix} + \begin{vmatrix} x & b & b \\ a & x & b \\ \frac{d}{dx}(a) & \frac{d}{dx}(a) & \frac{d}{dx}(x) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & x & b \\ a & a & x \end{vmatrix} + \begin{vmatrix} x & b & b \\ 0 & 1 & 0 \\ a & a & x \end{vmatrix} + \begin{vmatrix} x & b & b \\ a & x & b \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x & b \\ a & x \end{vmatrix} + \begin{vmatrix} x & b \\ a & x \end{vmatrix} + \begin{vmatrix} x & b \\ a & x \end{vmatrix} = 3 \begin{vmatrix} x & b \\ a & x \end{vmatrix} = 3\Delta_2.$$

Ans. [C]

Ex.11 The value of the determinant (when $n \in N$)

$$D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix} =$$

(A) $(n!)^3 ((2n^3 + 8n^2 + 10n + 4)$

(B) $(n!)^3 (2n^2 + 8n + 10)$

(C) $(n!)^2 (2n^3 + 8n^2 + 10n + 4)$

(D) none of these

Sol. Here $D = (n!)^3$

$$\begin{vmatrix} 1 & n+1 & (n+2)(n+1) \\ n+1 & (n+2)(n+1) & (n+3)(n+2)(n+1) \\ (n+2)(n+1) & (n+3)(n+2)(n+1) & (n+4)(n+3)(n+2)(n+1) \end{vmatrix}$$

$= (n!)^3 (n + 1)^2 (n + 2)$

$$\begin{vmatrix} 1 & 1 & 1 \\ n+1 & n+2 & (n+3) \\ (n+2)(n+1) & (n+3)(n+2) & (n+4)(n+3) \end{vmatrix}$$

operating $C_2 - C_1$, $C_3 - C_2$ and expanding

$= (n!)^3 (n +1)^2 (n + 2).\ 2$

$= (n!)^3((2n^3 + 8n^2 + 10n + 4)$ as on simplification.

Note : The answer may be verified by taking $n = 1$

Ans. [A]

Ex.12 If $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = Px - 12$ then-

(A) $P = 24$ (B) $P = -24$

(C) $P = 0$ (D) $P = 12$

Sol. Applying - $R_2 \to R_2 - (R_1 + R_3)$

Determinant $\begin{vmatrix} x^2+x & x+1 & x-2 \\ -4 & 0 & 0 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$

Applying $= R_1 \to R_1 + \frac{1}{4}x^2R_2$

and $R_3 \to R_3 + \frac{1}{4}x^2R_2$

Determinant $= \begin{vmatrix} x & x+1 & x-2 \\ -4 & 0 & 0 \\ 2x+3 & 2x-1 & 2x-1 \end{vmatrix}$

Applying $= R_3 \to R_3 - 2R_1$

Determinant $= \begin{vmatrix} x+0 & x+1 & x-2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$

$$= \begin{vmatrix} x & x & x \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$$

$$= x \begin{vmatrix} 1 & 1 & 1 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$$

$= (24x - 12) \Rightarrow P = 24.$ **Ans. [A]**

Ex.13 If $\Delta_r = \begin{vmatrix} r-1 & n & 6 \\ (r-1)^2 & 2n^2 & 4n-2 \\ (r-1)^3 & 3n^2 & 3n^2-3n \end{vmatrix}$, then

$\sum_{r=1}^{n} \Delta_r$ equals -

(A) 1 (B) -1

(C) 0 (D) None of these

Sol. $\because \sum_{r=1}^{n} (r-1) = 1 + 2 + \ldots.. + (n-1)$

$= \frac{n(n-1)}{2}$

$$\sum_{r=1}^{n}(r-1)^2 = 1^2 + 2^2 + ... + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}$$

$$\sum_{r=1}^{n}(r-1)^3 = 1^3 + 2^3 + ... + (n-1)^3$$

$$= \frac{n^2(n-1)^2}{4}$$

$$\therefore \sum_{r=1}^{n}\Delta_r \begin{vmatrix} \frac{n(n-1)}{2} & n & 6 \\ \frac{1}{6}n(n-1)(2n-1) & 2n^2 & 2(2n-1) \\ \frac{1}{4}n^2(n-1)^2 & 3n^3 & 3n(n-1) \end{vmatrix}$$

$$= \frac{n(n-1)}{12}\begin{vmatrix} 6 & n & 6 \\ 2(2n-1) & 2n^2 & 2(2n-1) \\ 3n(n-1) & 3n^3 & 3n(n-1) \end{vmatrix} = 0$$

Ans. [C]

Ex.14 If a, b, c, are p^{th}, q^{th} and r^{th} terms respectively of a H.P., then value of the determinant $\begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$ is -

(A) p + q + r (B) 0

(C) 1/p + 1/q + 1/r (D) None of these

Sol. Let A and d be the first term and common difference of the corresponding A.P., then

$$\frac{1}{a} = A + (p-1)d$$

$$\frac{1}{b} = A + (q-1)d$$

$$\frac{1}{c} = A + (r-1)d$$

$$\text{Now det.} = abc\begin{vmatrix} 1/a & 1/b & 1/c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

$$= abc \begin{vmatrix} A+(p-1)d & A+(q-1)d & A+(r-1)d \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

$$= abc\begin{vmatrix} A & A & A \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} + abcd\begin{vmatrix} p-1 & q-1 & r-1 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

= 0 + 0 = 0 **Ans. [B]**