FUNCTION

(KEY CONCEPTS + SOLVED EXAMPLES)

FUNCTION

1. *Numbers and their sets*
2. *Interval*
3. *Function*
4. *Classification of Functions*
5. *Value of the Function*
6. *Equal Function*
7. *Kinds of functions*
8. *Composite Functions*
9. *Inverse Functions*
10. *Domain & Range of Functions*
11. *Functions & their graphs*

KEY CONCEPTS

1. Numbers and their Sets

(a) Natural Numbers : N = {1,2,3,4,...}

(b) Whole Numbers : W = {0,1,2,3,4,....}

(c) Integer Numbers :

I or Z = {...–3,–2,–1, 0,1,2,3,...},

Z^+ = {1,2,3,....}, Z^- = {–1,–2,–3,....}

Z_0 = {± 1, ± 2, ± 3,....}

(d) Rational Numbers :

$Q = \{ \frac{p}{q} ; p, q \in z, q \neq 0 \}$

Ex. $\{1, \frac{5}{3}, -10, 10^5, \frac{22}{7}, \frac{-20}{3}, 0 ....\}$

Note :

(i) In rational numbers the digits are repeated after decimal.

(ii) 0 (zero) is a rational number.

(e) Irrational numbers: The numbers which are not rational or which can not be written in the form of p/q ,called irrational numbers

Ex. $\{ \sqrt{2}, \sqrt{3}, 2^{1/3}, 5^{1/4}, \pi, e, ...\}$

Note:

(i) In irrational numbers, digits are not repeated after decimal.

(ii) π and e are called special irrational quantities.

(iii) ∞ is neither a rational number nor a irrational number.

(f) Real Numbers : {x, where x is rational and irrational number}

Ex. $R = \{ 1, 1000, 20/6, \pi, \sqrt{2}, -10, -\frac{20}{3}, ...\}$

(g) Positive Real Numbers: $R^+ = (0,\infty)$

(h) Negative Real Numbers : $R^- = (-\infty,0)$

(i) R_0 : all real numbers except 0 (Zero).

(j) Imaginary Numbers : C = {i,ω,....}

(k) Prime Numbers :

These are the natural numbers greater than 1 which is divisible by 1 and itself only, called prime numbers.

2,3,5,7,11,13,17,19,23,29,31,37,41,...

(l) Even Numbers : E = {0,2,4,6,....}

(m) Odd Numbers : O = {1,3,5,7,....}

2. Interval

The set of the numbers between any two real numbers is called **interval**.

(a) Close Interval :

$[a,b] = \{ x, a \leq x \leq b \}$

(b) Open Interval:

(a, b) or $]a, b[= \{ x, a < x < b \}$

(c) Semi open or semi close interval:

$[a,b[$ or $[a,b) = \{x; a \leq x < b\}$

$]a,b]$ or $(a,b] = \{x ; a < x \leq b\}$

3. Function

Let A and B be two given sets and if each element a ∈ A is associated with a unique element b ∈ B under a rule f , then this relation is called **function**.

Here b, is called the image of a and a is called the pre- image of b under f.

Note :

(i) Every element of A should be associated with B but vice-versa is not essential.

(ii) Every element of A should be associated with a unique (one and only one) element of but any element of B can have two or more relations in A.

3.1 Representation of Function :

It can be done by three methods :

(a) By Mapping

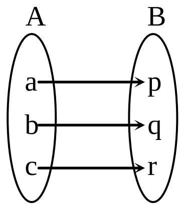
(b) By Algebraic Method

(c) In the form of Ordered pairs

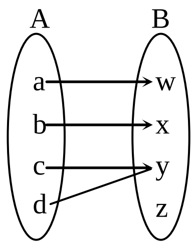
(A) Mapping :

It shows the graphical aspect of the relation of the elements of A with the elements of B .

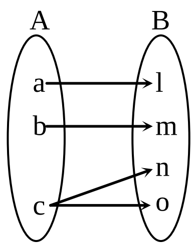
Ex. f_1**:**



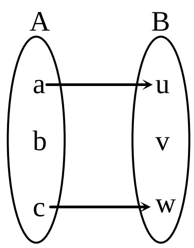
f_2**:**



f_3**:**



f_4**:**



In the above given mappings rule f_1 and f_2 shows a function because each element of A is associated with a unique element of B. Whereas f_3 and f_4 are not function because in f_3, element c is associated with two elements of B, and in f_4 , b is not associated with any element of B, which do not follow the definition of function. In f_2, c and d are associated with same element, still it obeys the rule of definition of function because it does not tell that every element of A should be associated with different elements of B.

(B) Algebraic Method :

It shows the relation between the elements of two sets in the form of two variables x and y where x is independent variable and y is dependent variable.

If A and B be two given sets

A = { 1,2,3 }, B = {5,7,9}

then $f : A \to B$, $y = f(x) = 2x + 3$.

(C) In the form of ordered pairs :

A function $f : A \to B$ can be expressed as a set of ordered pairs in which first element of every ordered pair is a member of A and second element is the member of B. So f is a set of ordered pairs (a, b) such that :

(i) a is an element of A

(ii) b is an element of B

(iii) Two ordered pairs should not have the same first element.

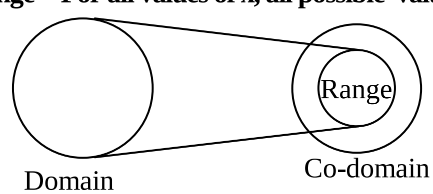
3.2 Domain, Co-domain and Range :

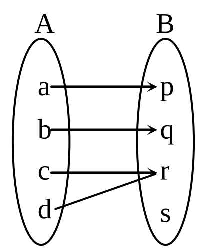
If a function f is defined from a set of A to set B then for $f: A \to B$ set A is called the **domain** of function f and set B is called the **co-domain** of function f. The set of the f- images of the elements of A is called the **range** of function f.

In other words, we can say

Domain = All possible values of x for which f(x) exists.

Range = For all values of x, all possible values of f(x).





Domain = {a,b,c,d} = A

Co-domain = {p,q,r,s} = B

Range = {p,q,r}

3.3 Algebra of functions:

Let f and g be two given functions and their domain are D_f and D_g respectively, then the sum, difference, product and quotient functions are defined as :

(a) $(f + g)(x) = f(x) + g(x), \forall x \in D_f \cap D_g$

(b) $(f - g)(x) = f(x) - g(x), \forall x \in D_f \cap D_g$

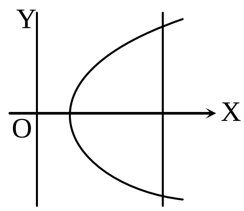
(c) $(f . g)(x) = f(x) . g(x), \forall x \in D_f \cap D_g$

(d) $(f/g)(x) = \dfrac{f(x)}{g(x)}; g(x) \neq 0, \forall x \in D_f \cap D_g$

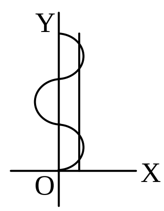
3.4 Testing for a function : A relation f : A B is a function or not, it can be checked by following methods.

(a) See Article 3 (a) & 3 (b)

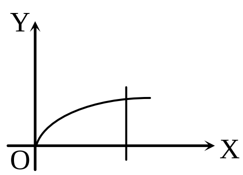
(b) Vertical Line Test : If we are given a graph of the relation then we can check whether the given relation is function or not . If it is possible to draw a vertical line which cuts the given curve at more than one point then given relation is not a function and when this vertical line means line parallel to Y - axis cuts the curve at only one point then it is a function.



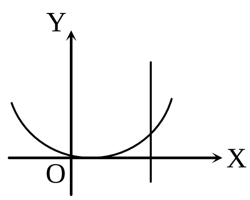
(i)



(ii)



(iii)



(iv)

fig. (iii) and (iv) represents a function.

4. Classification of Function

4.1 Algebric and Transcendental function :

4.1.1 Algebric Function :

The function which consists of sum, difference, product, quotient, power or roots of a variable is called the algebric function.

Algebric functions further can be classified as-

(a) Polynomial or integral Function :

The function of the form

$f(x) = a_0 x^n + a_1 x^{n-1} + ...+ a_{n-1}x + a_n$, $a_0 \neq 0$

Where a_0 , a_1 , a_2 , ...a_n are constants and $n \in N$ is called polynomial or Integral Functions.

(i) f(x) = C, is a polynomial of zero power or a constant function.

(ii) f(x) = a x + b, is a polynomial of power one or a linear function.

(iii) $f(x) = ax^2 + bx + c$, is a polynomial of two power or a quadratic function and so on.

(b) Rational Function :

The quotient of two polynomial functions is called the Rational function.

(c) Irrational Function:

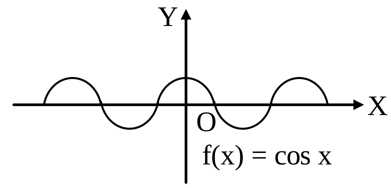
A function which is not rational is called Irrational Function.

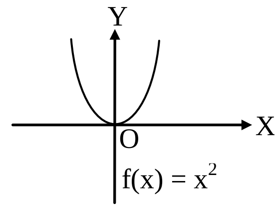
4.1.2 Transcendental Function: The function which is not algebric is called transcendental function.

4.2 Even or Odd Function:

4.2.1 Even function :

If we put (–x) in place of x in the given function and if f(–x) = f(x) , $x \in$ domain then function f(x) is called even function.





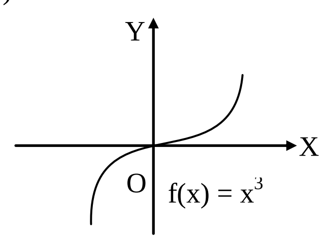
Note :

The graph of even function is always symmetric with respect to y-axis.

4.2.2 Odd function :

If we put (–x) in place of x in the given function and if f (–x) = – f(x), $\forall x \in$ domain then f (x) is called odd function.

Then f(x) is called odd function.



Note :

The graph of odd function is always symmetric with respect to origin.

Properties of Even and Odd Function :

(a) The product of two even functions is even function.

(b) The sum and difference of two even functions is even function.

(c) The sum and difference of two odd functions is odd function.

(d) The product of two odd functions is even function.

(e) The product of an even and an odd function is odd function.

(f) The sum of even and odd function is neither even nor odd function.

(g) It is not essential that every function is even or odd. It is possible to have some functions which are neither even nor odd function.

Note :

Zero function $f(x) = 0$ is the only function which is even and odd both.

4.3 Explicit and Implicit function :

(a) Explicit Function :

A function is said to be explicit if it can be expressed directly in terms of the independent variable.

$y = f(x)$ or $x = (y)$

(b) Implicit Function :

A function is said to be implicit if it can not be expressed directly in terms of the independent variable.

4.4 Continuous and Discontinuous Function :

(a) Continuous Function :

A function is said to be continuous function in an interval I if we are not required to lift the pen or pencil off the paper i.e. there is no gap or break or jump in the graph.

(b) Discontinuous Function :

A function is said to be discontinuous if there is a break or gap or jump in the graph of the function at any point.

4.5 Increasing and Decreasing Function :

(a) Increasing Function :

A function f(x) is called increasing function in the domain D if the value of the function does not decrease by increasing the value of x.

so $x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2)\ \forall\ x_1, x_2 \in$ domain

or $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)\ \forall\ x_1, x_2 \in$ domain

A function is called strictly increasing if

if $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$

or $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)\ \forall\ x_1, x_2 \in$ domain

(b) Decreasing Function :

A function f (x) is said to be decreasing function in the domain D if the value of the function does not increase by increasing the value of x (variable).

so if $x_1 > x_2 \Rightarrow f(x_1) \leq f(x_2)$

or $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)\ \forall\ x_1, x_2 \in D$

A function is called strictly decreasing if

if $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$

or $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)\ \forall\ x_1, x_2 \in D$

Note :

It is not essential for any function to be increasing or decreasing. There are some functions which are neither increasing nor decreasing i.e. function is increasing in one part of given interval and decreasing in second part.

4.6 Greatest Integer Function :

A function is said to be greatest integer function if it is of the form of f(x) = [x] where [x] = integer equal or less than x.

$f(x) = y = [x]$

$0 \leq x < 1 \Rightarrow y = 0$

$1 \leq x < 2 \Rightarrow y = 1$

$2 \leq x < 3 \Rightarrow y = 2$

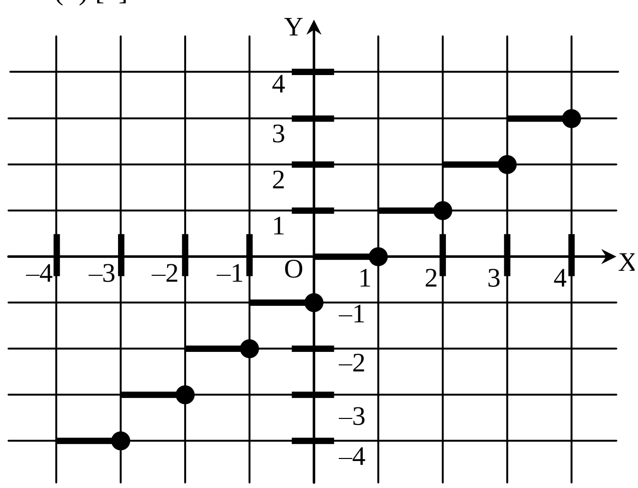
⋮

and so on

Note : Important Identities :

(i) $[x] \leq x$ **(** This is always true**)**

(ii) $[x] + 1 > x$



4.7 Periodic Function:

A function is said to be periodic function if its each value is repeated after a definite interval.

So a function f(x) will be periodic if a positive real number T exist such that,

$f(x + T) = f(x),\ \forall\ x \in$ Domain

Here the least positive value of T is called the period of the function. Clearly $f(x) = f(x+T) =$

$f(x + 2T) = f(x + 3T) = ...$

For example, sin x, cos x, tan x are periodic functions with period 2π, 2π & π respectively.

Note:

(a) If function f (x) has period T then

$$f(nx) \text{ has period } \frac{T}{n}$$

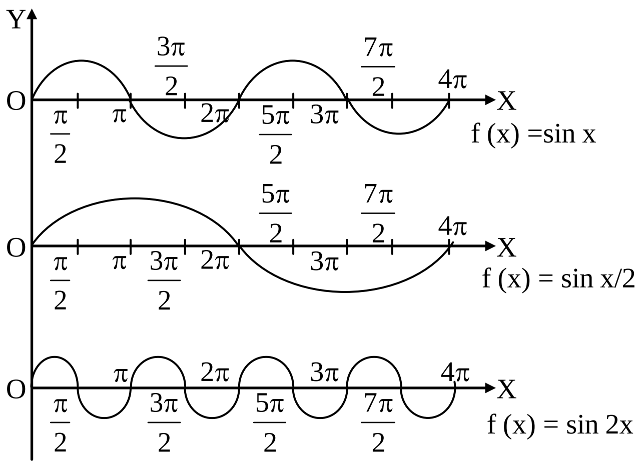
$$f(x/n) \text{ has period } nT.$$

$$f(ax + b) \text{ has period } \frac{T}{|a|}$$

(b) If the period of f (x) and g (x) are same (T) then the period of a f (x) + bg (x) will also be T.

(c) If the period of f(x) is T_1 and g (x) has T_2, then the period of $f(x) \pm g(x)$ will be LCM of T_1 and T_2 provided it satisfies the definition of periodic function.

The graphs of f(x) = sin x, f(x) = sin x/2, and f(x) = sin 2x are being compared to find the period.



5. Value of the Function

If y = f(x) is any function defined in R, then for any given value of x (say x = a), the value of the function f(x) can be obtained by substituting x = a in it and it is denoted by f(a).

6. Equal Function

Two functions $f : A \to B$ and $g : C \to D$ are called equal functions if and only if

(a) domain of f = domain of g

(b) co-domain of f = co-domain of g

(c) $f(x) = g(x)$, $\forall\, x \in$ domain

7. Kinds of Functions

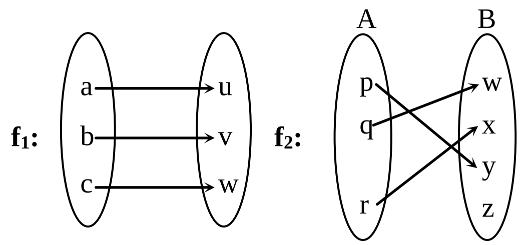
7.1 One - One function or Injection :

A function $f : A \to B$ is said to be one- one if different elements of A have different images in B. Therefore for any two elements x_1, x_2 of a set A,

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

or $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

then function is one-one.

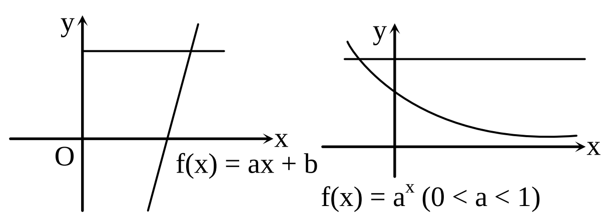


The above given diagrams f_1 & f_2 shows one-one function.

Note :

(a) If function is given in the form of ordered pairs and if no two ordered pairs have same second element then function is one-one.

(b) If the graph of the function y = f(x) is given, and each line parallel to x-axis cuts the given curve at maximum one point then function is one-one.



Examples of One-One Function –

(i) $f : R \to R$, $f(x) = x$,

(ii) $f : R \to R$, $f(x) = ax + b$

(iii) $f : R \to R$, $f(x) = ax^n + b$, n is odd positive integer

(iv) $f : R \to R$, $f(x) = x\,|x|$

(v) $f : R \to R$, $f(x) = e^x$,

(vi) $f : R \to R$, $f(x) = a^x$ (a > 0)

(vii) $f : R \to R$, f(x) = sinh x,

(viii) $f : R \to R$, f(x) = tanh (x)

(ix) $f : R_0 \to R$, $f(x) = 1/x$,

(x) $f : R^+ \to R$, f(x) = log x ,

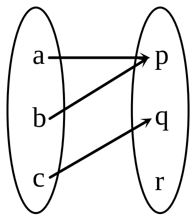
(xi) $f : R_0 \to R$, $f(x) = \log_a x$ (a>0)

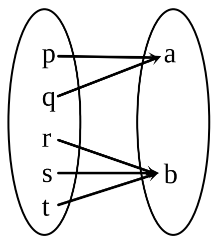
7.2 Many-one Function :

A function f : A → B is called many- one, if two or more different elements of A have the same f-image in B.

therefore f : A B is many-one if

$x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$

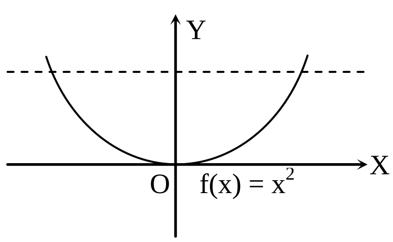


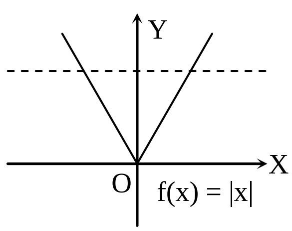


The above given arrow-diagrams show many-one function.

(a) If function is given in the form of set of ordered pairs and the second element of atleast two ordered pairs are same then function is many- one.

(b) If the graph of y = f(x) is given and the line parallel to x-axis cuts the curve at more than one point then function is many-one.





Example of many-one function :

(i) $f : R \to R$, f(x) = C, where C is a constant

(ii) (ii) $f : R \to R$, $f(x) = x^2$

(iii) $f : R \to R$, $f(x) = ax^2 + b$,

(iv) $f : R \to R$, $f(x) = |x|$,

(v) $f : R \to R$, $f(x) = x + |x|$

(vi) $f : R \to R$, $f(x) = x - |x|$

(vii) $f : R \to R$, f(x) = cosh x

(viii) $f : R \to R$, f(x) = [x],

(ix) $f : R \to R$, f(x) = x – [x]

Where [x] is greatest integer function.

Methods to check trigonometrical functions to be one-one or many-one.

(a) If the domain of the function is in one quadrant then trigonometrical functions are always one-one.

(b) If trigonometrical function changes its sign in two consecutive quadrants then it is one-one but if it does not change the sign then it is many one.

$f : (0, \pi)$, f (x) = sin x many-one

and $f : (0, \pi)$, f (x) = cos x one-one

(c) In three consecutive quadrants trigonometrical functions are always one-one.

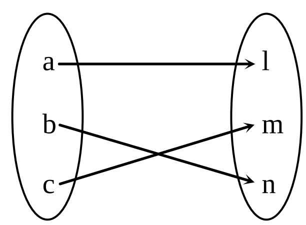
7.3 Onto function or Surjection :

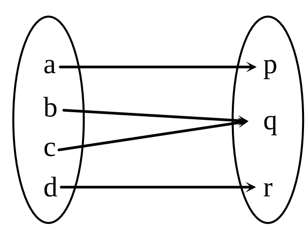
A function f : A → B is onto if the each element of B has its pre- image in A. Therefore if $f^{-1}(y) \in A$, $\forall\, y \in B$ then function is onto.

In other words.

Range of f = Co-domain of f.

The following arrow-diagram shows onto function.





Examples of onto function :

(i) $f : R \to R$, f(x) = x,

(ii) $f : R \to R$, f(x) = ax + b, $a \neq 0$, $b \in R$

(iii) $f : R \to R$, $f(x) = x^3$

(iv) $f : R \to R$, f(x) = x|x|

(v) $f : R \to R^+$, $f(x) = e^x$

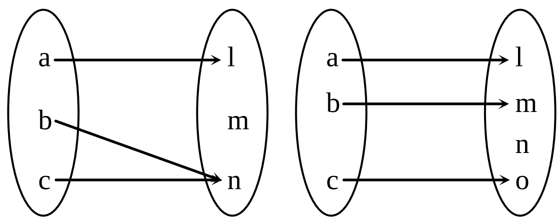
(vi) $f : R^+ \to R$, f(x) = log x.

7.4 Into function :

A function f : A $\to$ B is into if there exist atleast one element in B which is not the f-image of any element in A. Therefore, atleast one element of B such that $f^{-1}(y) = \phi$ then function is into. In other words

Range of f $\neq$ co-domain of f

The following arrow-diagram shows into function.



Examples of into function :

(i) f : R$\to$ R, $f(x) = x^2$

(ii) f : R $\to$R, f(x) = |x|

(iii) f : R $\to$ R, f(x) = c (c is constant)

(iv) f : R $\to$ R, f(x) = sin x

(v) f : R $\to$ R, f(x) = cos x

(vi) f : R $\to$ R, $f(x) = e^x$

(vii) f : R $\to$ R, $f(x) = a^x$, a > 0

Note :

For a function to be onto or into depends mainly on their co-domain.

Now, we can classify the function further into four categories :

7.5 One-one onto function or bijection :

A function f is said to be one-one onto if f is one-one and onto both.

7.6 One-one into function :

A function is said to be one- one into if f is one-one but not onto.

7.7 Many one-onto function :

A function f : A $\to$ B is many one-onto if f is onto but not one one.

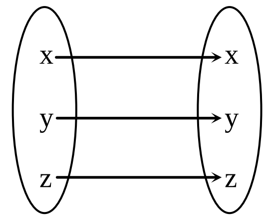
7.8 Many one-into function :

A function is said to be many one-into if it is neither one-one nor onto.

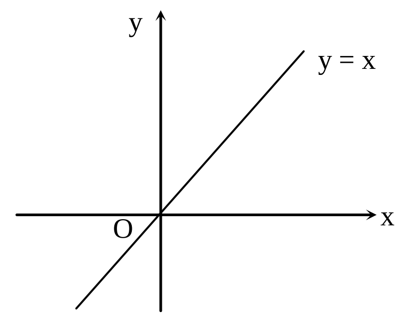
7.9 Identity Function :

Let A be any set and the function f : A$\to$ A be defined as f(x) = x , $\forall$ x $\in$ A i.e. if each element of A is mapped by itself then f is called the identity function . It is represented by I_A.

If A = {x, y,z} then I_A = {(x,x), (y,y), (z,z)}



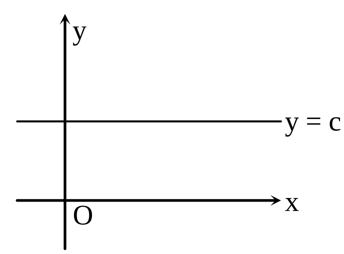
Graph -

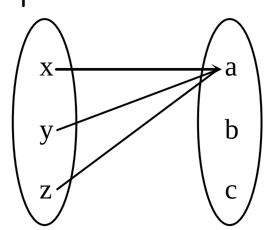


Domain = R
Range = R

7.10 Constant function :

If in a function f : A $\to$ B f(a) = c, $\forall$ a $\in$ A. then it is a constant function.



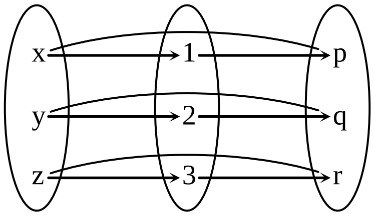


The range of constant function contains only one element.

8. Composite Function

If f : A$\to$ B and g : B$\to$ C are two function then the composite function of f and g , gof : A $\to$ C will be defined as

gof(x) = g[f(x)], $\forall$ x $\in$ A.



Note :

(a) Function gof will exist only when range of f is the subset of domain of g.

(b) gof (x) is simply the g-image of f(x), where f(x) is f-image of elements $x \in A$.

(c) fog does not exist here because range of g is not a subset of domain of f.

Properties of composite function :

(a) If f and g are two functions then for composite of two functions $fog \neq gof$.

(b) Composite functions obeys the property of associativity i.e. fo (goh) = (fog) oh.

(c) Composite function of two one-one onto functions if exist, will also be a one-one onto function.

9. Inverse Function

If $f : A \to B$ be a one-one onto (bijection) function, then the mapping $f^{-1} : B \to A$ which associates each element $b \in B$ with element a $\forall$ A, such that f(a) = b, is called the **inverse function** of the function $f : A \to B$

$f^{-1} : B \to A,\ f^{-1}(b) = a \Rightarrow f(a) = b$

In terms of ordered pairs inverse function is defined as -

$f^{-1} = \{(b,a) \mid (a,b) \in f\}$

Note :

For the existence of inverse function, it should be one-one and onto.

Properties :

(a) Inverse of a bijection is also a bijection function.

(b) Inverse of a bijection is unique.

(c) $(f^{-1})^{-1} = f$

(d) If f and g are two bijections such that (gof) exists then $(gof)^{-1} = f^{-1}og^{-1}$

(e) If $f : A \to B$ is a bijection then $f^{-1}: B \to A$ is an inverse function of f.

$f^{-1}of = I_A$ and $fof^{-1} = I_B$.

Here I_A, is an identity function on set A, and I_B, is an identity function on set B.

10. Domain & Range of Some Standard Function

Function	**Domain**	**Range**
Polynomial function	R	R
Identity function x	R	R
Constant function c	R	{c}
Reciprocal fn 1/x	R_0	R_0
Signum function	R	{–1,0,1}
$ax + b$; $a, b \in R$	R	R
$ax^3 + b$; $a, b \in R$	R	R
x^2, $\|x\|$	R	$R^+ \cup \{0\}$
x^3, $x\|x\|$	R	R
$x + \|x\|$	R	$R^+ \cup \{0\}$
$x - \|x\|$	R	$R^- \cup \{0\}$
[x]	R	Z
x – [x]	R	[0,1)
$\|x\|/ x$	R_0	{–1,1}
$\sqrt{x}$	$[0, \infty)$	$[0, \infty)$
a^x	R	R^+
log x	R^+	R
sin x	R	[–1,1]
cos x	R	[–1,1]
tan x	$R - \{(2n + 1)\pi/2 \mid n \in z\}$	R
cot x	$R - \{n\pi \mid n \in z\}$	R
sec x	$R - \{(2n + 1)\pi/2 \mid n \in z\}$	R – (–1,1)
cosec x	$R - \{n\pi \mid n \in z\}$	R – (–1,1)
$\sin^{-1} x$	[–1,1]	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	[–1,1]	$[0, \pi]$
$\tan^{-1} x$	R	$(-\pi/2, +\pi/2)$
$\cot^{-1} x$	R	$(0,\pi)$
$\sec^{-1} x$	R – (–1,1)	$[0,\pi] - \{\pi/2\}$

$\text{cosec}^{-1} x$	$R - (-1,1)$	$(-\pi/2, \pi/2] - \{0\}$

11. Some Functions & Their Graphs

11.1 Classification of function :

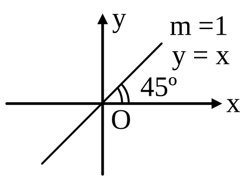
- Function
  - Algebric
    - Polynomial
    - Rational
    - Irrational
    - Piecewise
      - Modulus
      - Greatest integer function
      - Fractional part
      - Signum
  - Transcendental
    - Trigonometric
    - Exponential
    - Logarithmic
    - Inverse Trigonometric

Graphs of Function :

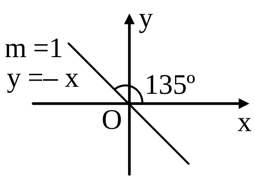
(1) Polynomials

(A) $y = f(x) = mx$, where m is a constant $(m \neq 0)$

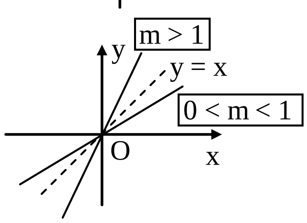
(i)



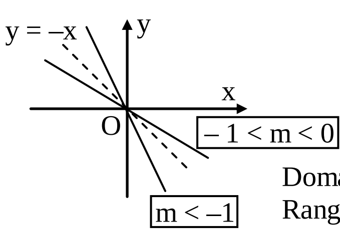
(ii)



(iii)



(iv)

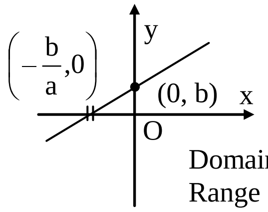


Domain = R

Range = R

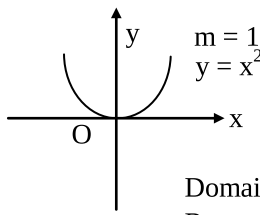
$y = mx$ is coutinuous function & one-one function $(m \neq 0)$

(v) $y = ax + b$, where a & b are constant

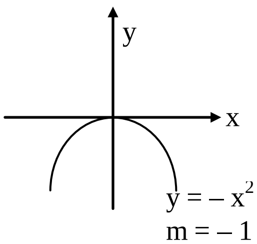


Domain = R
Range = R

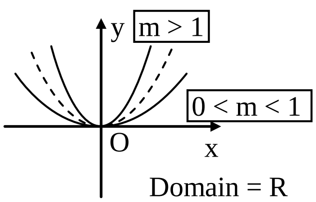
(B) $y = mx^2$, $(m \neq 0)$, $x \in R$



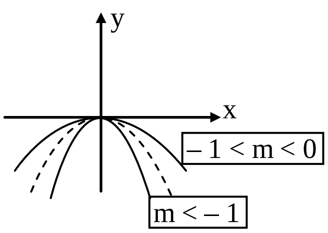
Domain = R
Range = $[0, \infty]$



Domain = R
Range = $(\infty, 0]$



Domain = R
Range = $[0, \infty)$

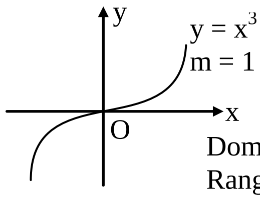


Domain = R
Range = $[0, \infty)$

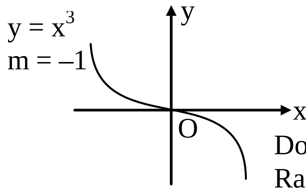
$y = mx^2$ is continuous, even function and many-one function.

Graph is symmetric about y-axis

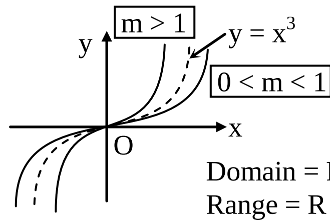
(C) $y = mx^3$, $(m \neq 0)$, $x \in R$



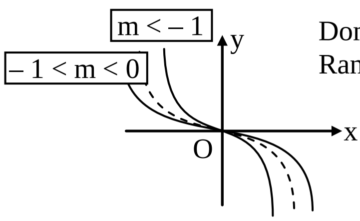
Domain = R
Range = R



Domain = R
Range = R



Domain = R
Range = R

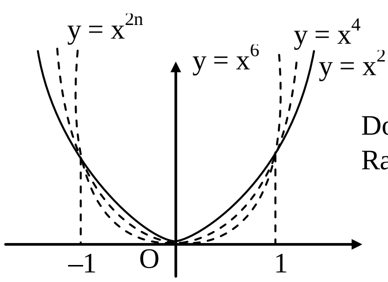


Domain = R
Range = R

$y = mx^3$ is continuous, odd one-one & increasing function.

Graph is symmetric about origin.

(D) $y = x^{2n}, n \in N$



Domain = R
Range = $[0, \infty)$

$f(x) = x^{2n}$ is an even, continuous function. It is many one function.

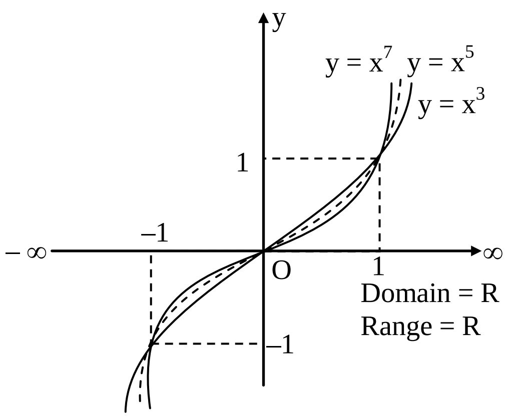
From the graph :

For $|x| \le 1 \Rightarrow x^6 \le x^4 \le x^2$

For $|x| > 1 \Rightarrow x^6 > x^4 > x^2$

Graph is symmetric about y-axis

(E) $y = x^{2n+1}, n \in N$



$f(x) = x^{2n+1}$, $n \in N$ is an odd continuous function. It is always one-one function.

From the graph :

For $x \in (-\infty, -1] \cup [0, 1]$

$\Rightarrow x^3 \geq x^5 \geq x^7$

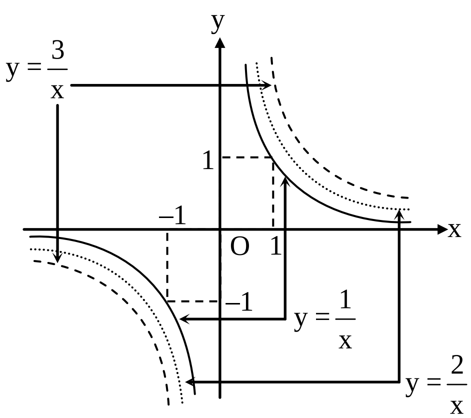
For $x \in (-1, 0) \cup (1, \infty)$

$\Rightarrow x^3 < x^5 < x^7$

Graph is symmetric about origin.

(2) Rational Function :

(A) $y = \dfrac{m}{x}$, $x \neq 0$, $m \neq 0$



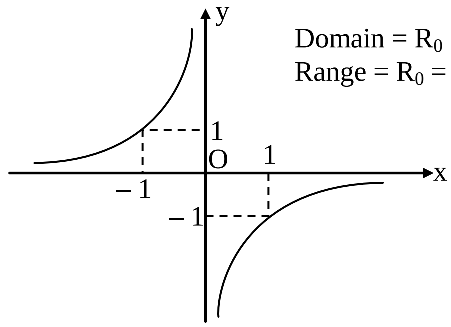
$y = \dfrac{m}{x}$ is odd function discontinuous at x = 0.

Domain = $R_0 = R - \{0\}$

Range = $R_0 = R - \{0\}$

This type of graph xy = m is called rectangular hyperbola.

(B) $y = -\dfrac{1}{x}$, $x \neq 0$



Domain = R_0 = R – {0}
Range = R_0 = R – {0}

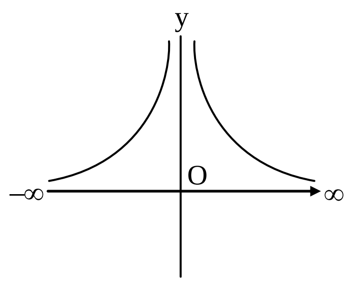
$y = -\frac{1}{x}$ is an odd & one-one function discontinuous at x = 0

(C) $y = \frac{1}{x^2}$, $x \neq 0$

$y = \frac{1}{x^2}$ is even function, many one and discontinuous at x = 0

Domain = R_0

Range = $R^+ = (0, \infty)$



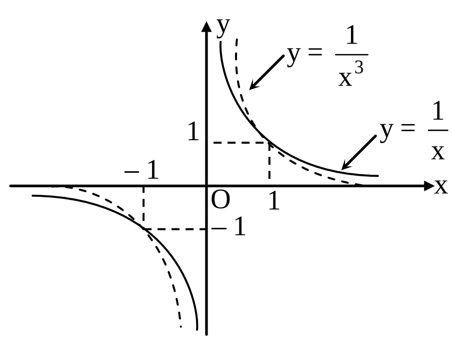
(D) $y = \frac{1}{x^{2n-1}}$, $n \in N$

An odd one-one function discontinuous at x = 0

From graph

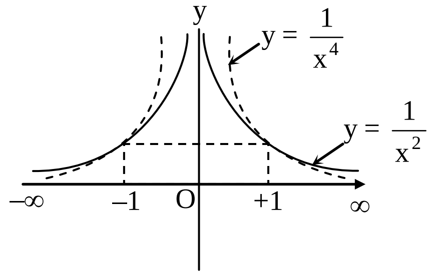
$0 < x < 1 \Rightarrow \frac{1}{x^3} > \frac{1}{x}$

$x > 1 \Rightarrow \frac{1}{x^3} < \frac{1}{x}$.



(E) $y = \frac{1}{x^{2n}}$, $n \in N$

$y = \dfrac{1}{x^{2n}}$ is an even function & many one, discontinuous at x = 0.



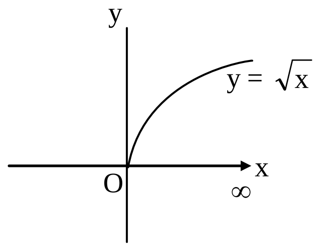
Domain = R_0
Range = R^+

(3) Irrational function

(A) $y = \sqrt{x} = x^{1/2}$

This function is neither even nor odd, one-one and continuous function in it's domain.

It is an increasing function.

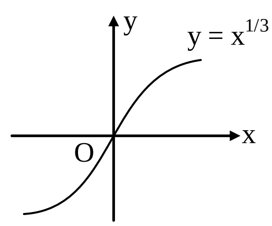


Domain = $[0, \infty)$

Range = $[0, \infty)$

(B) $y = \sqrt[3]{x} = x^{1/3}$

This function is an odd, continuous and one-one function. It is an increasing function.



Domain = R
Range = R

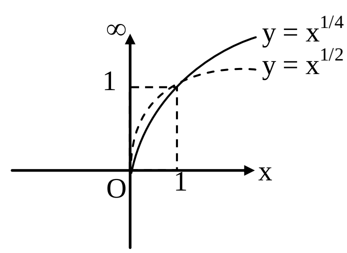
(C) $y = x^{1/2n}$, $n \in N$

This function is neither even nor odd, continuous and increasing function it is also one-one function

From graph

$0 < x < 1 \Rightarrow > \sqrt[2]{x} > \sqrt[4]{x}$

$x > 1 \Rightarrow x^{1/2} < x^{1/4}$



Domain = $R^+ = [0, \infty)$
Range = $R^+ = [0, \infty)$

(D) $y = x^{\frac{1}{2n+1}}$, $n \in N$

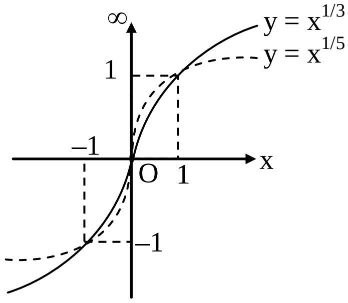
This function is an odd, continuous, one-one and increasing function

From graph

$x \in (-\infty, -1) \cup (0, 1)$, $x^{1/3} > x^{1/5}$

$x \in (-1, 0) \cup (1, \infty)$, $x^{1/3} < x^{1/5}$

Graph is symmetric about origin.



Domain = R

Range = R

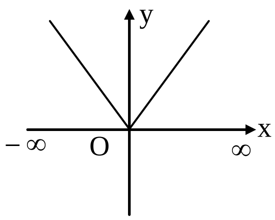
(4) Piecewise function (Special Function)

(A) Modulus function

$$y = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

This is an even, continuous, many one function.

Graph is symmetric about y-axis

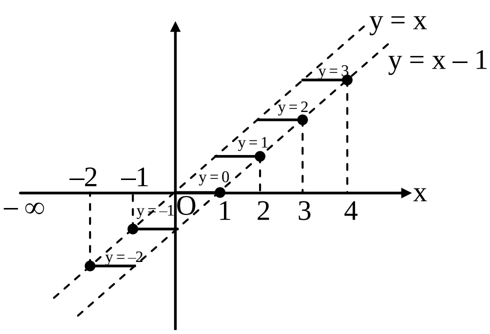


Domain = R

Range = $[0, \infty)$

(B) Greatest integer Function

[x] indicates the integral part of x which is nearest and smaller to x. It is also called step function.



$$y = [x] = \begin{cases} -1, & -1 \leq x < 0 \\ 0, & 0 \leq x < 1 \\ +1, & 1 \leq x < 2 \\ +2, & 2 \leq x < 3 \\ +3, & 3 \leq x < 4 \end{cases}$$

$\Rightarrow y = [x] = k, k \leq x < k + 1, k \in I$

Important Result from graph

$x - 1 < [x] \le x$

(C) Fractional part of x

$y = \{x\}$

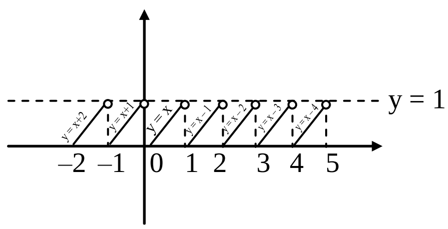
Any number is sum of integral & fractional part i.e. $x = [x] + \{x\}$

$\therefore\ y = \{x\} = x - [x] = x - k,\ k \le x < k + 1$

$$y = \{x\} = \begin{cases} x+1, & \ldots\ldots. \\ x, & 0 \le x < 1 \\ x-1, & 1 \le x < 2 \\ x-2, & 2 \le x < 3 \\ \ldots\ldots., & \ldots\ldots. \end{cases}$$

Domain = R

Range = [0, 1)

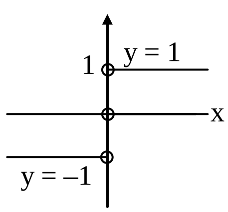


From graph $0 \le \{x\} < 1$

This function is a periodic function with period 1. This is also many-one function discontinuous at $x \in I$.

(D) $y = \text{sgn}(x) = \begin{cases} \dfrac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases} = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}$

This is odd, many-one function discontinuous at x = 0

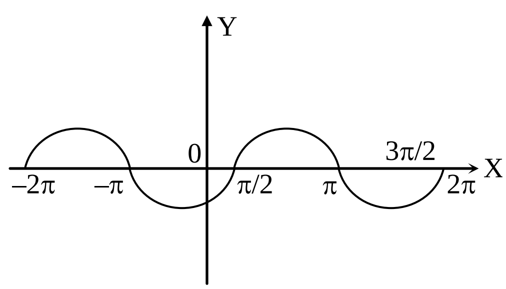


Domain = R

Range = {–1, 0, 1}

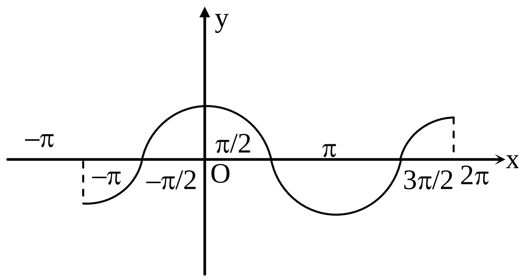
(5) Trigonometrical function :

(i) f (x) = sin x.



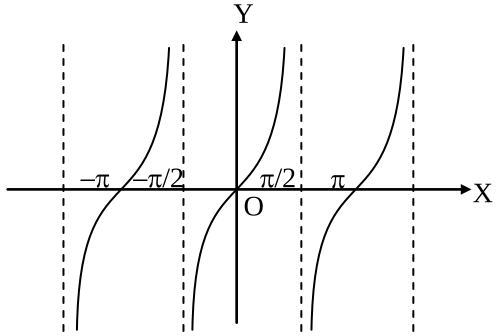
It is a continuous and many one function with period 2π.

(ii) f(x) = cos x



It is a continuous and many-one function having a period 2.π

(iii) f(x) = tan x



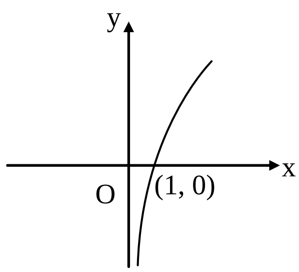
Domain = R – (2n+1) π/2, n ∈ I

Range = R

It is a discontinuous function at x = (2n+1)π/2, n ∈ I and periodic with period π.

(6) Logarithmic function :

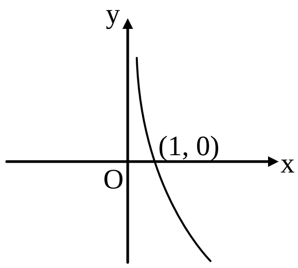
(i) f (x) = $\log_a x$ (a > 1)



Domain = R^+
Range = R

It is a continuous and one-one function.

(ii) f (x) = $\log_a x$ (a<1)

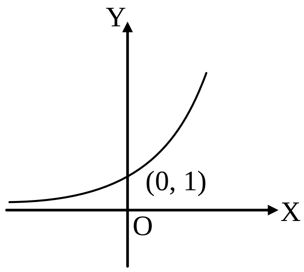


Domain = R^+
Range = R

It is a continuous and one-one function.

(74)Exponential function :

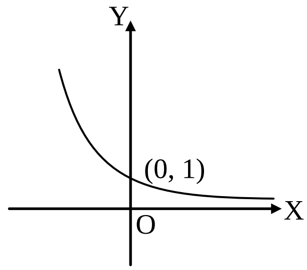
(i) f (x) = a^x (a>1)



Domain = R
Range = R^+

It is a continuous and one-one function.

(ii) $f(x) = a^x \quad (a<1)$

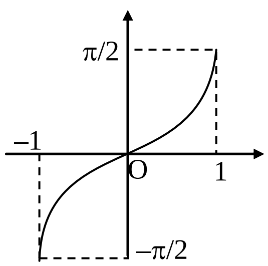


Domain = R
Range = R^+

It is a continuous and one-one function.

(8) Inverse Trigonometric Function :

$y = \sin^{-1} x$

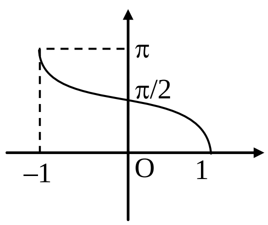


Domain = [–1, 1]

Range $= \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

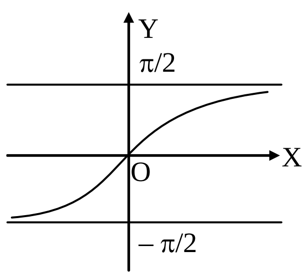
Odd function

$y = \cos^{-1}x$



Domain = [–1, 1]
Range = [0, π]
Neither even nor odd

$y = \tan^{-1}x$

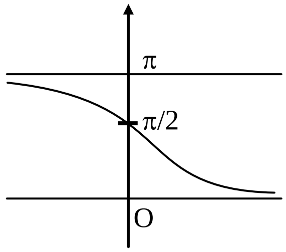


Domain = R

Range $= \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

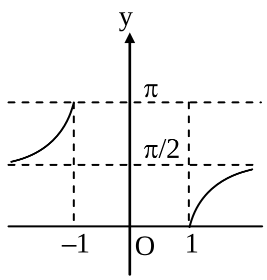
Odd function

$y = \cot^{-1} x$



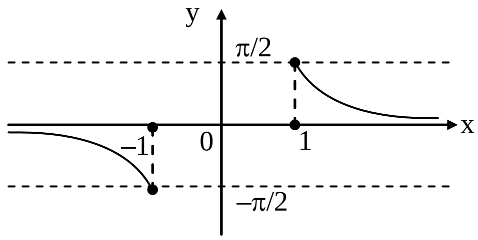
Domain = R
Range = $(0, \pi)$
Neither even nor odd

y = sec^{-1}x



Domain = $(-\infty, -1] \cup [1, \infty)$
Range = $[0, \pi] - \left\{\frac{\pi}{2}\right\}$
Neither even nor odd

y = cosec^{-1}x



Domain = $(-\infty, -1] \cup [1, \infty)$

Range = $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Odd function

All inverse trigonometric functions are monotonic.

SOLVED EXAMPLES

Ex.1 Which of the following is a function?

(A) {(2,1), (2,2), (2,3), (2,4)}

(B) {(1,4), (2,5), (1,6) , (3,9)}

(C) {(1,2), (3,3), (2,3), (1,4)}

(D) { (1,2), (2,2), (3,2), (4,2)}

Sol. We know that for a relation to be function every element of first set should be associated with one and only one element of second set but elements of first set can have same f-image in second set which is given in (D).

Ans.[D]

Ex.2 If $f(x) = \frac{x}{x-1} = \frac{1}{y}$, then f (y) equals

(A) x (B) x – 1

(C) x + 1 (D) 1 – x

Sol. $f(y) = \frac{y}{y-1} = \frac{(x-1)/x}{\frac{x-1}{x}-1} = \frac{x-1}{x-1-x} = 1 - x.$

Ans.[D]

Ex.3 The domain of $f(x) = \frac{1}{x^3 - x}$ is -

(A) R – {–1,0,1} (B) R

(C) R – {0,1} (D) None of these

Sol. Domain = $\{x;\ x \in R;\ x^3 - x \neq 0\}$

= R – {–1, 0,1}

Ans.[A]

Ex.4 The range of $f(x) = \cos \frac{\pi[x]}{2}$ is -

(A) {0,1} (B) {–1,1}

(C) {–1,0,1} (D) [–1,1]

Sol. [x] is an integer, cos (–x) = cos x and

$\cos\left(\frac{\pi}{2}\right) = 0,\ \cos 2\left(\frac{\pi}{2}\right) = -1.$

$\cos 0\left(\frac{\pi}{2}\right) = 1,\ \cos 3\left(\frac{\pi}{2}\right) = 0, ...$

Hence range = {–1,0,1}

Ans.[C]

Ex.5 If $f : R^+ \to R^+,\ f(x) = x^2 + 2$ and

$g : R^+ \to R^+,\ g(x) = \sqrt{x+1}$

then (f + g) (x) equals -

(A) $\sqrt{x^2+3}$ (B) x + 3

(C) $\sqrt{x^2+2} + (x+1)$ (D) $x^2 + 2 + \sqrt{(x+1)}$

Sol. (f + g) (x) = f(x) + g(x)

$= x^2 + 2 + \sqrt{x+1}$ **Ans. [D]**

Ex.6 Function $f(x) = x^{-2} + x^{-3}$ is -

(A) a rational function

(B) an irrational function

(C) an inverse function

(D) None of these

Sol. $f(x) = \frac{1}{x^2} + \frac{1}{x^3} = \frac{x+1}{x^3}$

= ratio of two polynomials

∴ f(x) is a rational function. **Ans.[A]**

Ex.7 The period of | sin 2x | is-

(A) π/4 (B) π/2 (C) π (D) 2π

Sol. Here $|\sin 2x| = \sqrt{\sin^2 2x}$

$= \sqrt{\frac{(1-\cos 4x)}{2}}$

Period of cos 4 x is π/2

Period of |sin 2x | will be π/2. **Ans.[B]**

Ex.8 If $f(x) = \frac{x-3}{x+1}$, then f [f {f (x)}] equals -

(A) x (B) 1/x (C) –x (D) –1/x

Sol. Here $f\{f(x)\} = f\left(\frac{x-3}{x+1}\right) = \frac{\left(\frac{x-3}{x+1}\right)-3}{\left(\frac{x-3}{x+1}\right)+1} = \frac{x+3}{1-x}$

$$\therefore f[f\{f(x)\}] = \frac{\frac{x+3}{1-x}-3}{\frac{x+3}{1-x}+1} = \frac{4x}{4} = x$$

Ans. [A]

Ex.9 If $f(x) = 2|x-2| - 3|x-3|$, then the value of f(x) when $2 < x < 3$ is -

(A) $5 - x$ (B) $x - 5$

(C) $5x - 13$ (D) None of these

Sol. $2 < x < 3 \Rightarrow |x-2| = x - 2$

$|x-3| = 3 - x$

$f(x) = 2(x-2) - 3(3-x) = 5x - 13.$

Ans. [C]

Ex.10 Which of the following functions defined from R to R are one-one -

(A) $f(x) = |x|$ (B) $f(x) = \cos x$

(C) $f(x) = e^x$ (D) $f(x) = x^2$

Sol. $x_1 \neq x_2 \Rightarrow e^{x_1} \neq e^{x_2}$

$\Rightarrow f(x_1) \neq f(x_2)$

$\therefore f(x) = e^x$ is one-one. **Ans. [C]**

Ex.11 The function $f : R \to R$, $f(x) = x^2$ is -

(A) one-one but not onto

(B) onto but not one-one

(C) one-one onto

(D) None of these

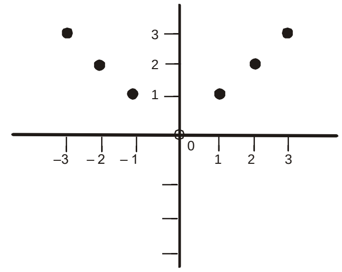
Sol. $\because\ 4 \neq -4$, but $f(4) = f(-4) = 16$

$\therefore$ f is many one function.

Again $f(R) = R^+ \cup \{0\}$ R, therefore f is into.

Ans. [D]

Ex.12 If $f : I_0 \to N$, $f(x) = |x|$, then f is -



(A) one-one (B) onto

(C) one-one onto (D) none of these

Sol. Observing the graph of this function, we find that every line parallel to x-axis meets its graph at more than one point so it is not one-one.

Now range of f = N = Co-domain, so it is onto.

Ans. [B]

Ex.13 If $f : R - \{3\}\ R - \{1\}$, $f(x) = \frac{x-2}{x-3}$ then function f(x) is -

(A) Only one-one (B) one-one into

(C) Many one onto (D) one-one onto

Sol. $\because\ f(x) = \frac{x-2}{x-3}$

$\therefore\ f'(x) = \frac{(x-3).1-(x-2).1}{(x-3)^2} = \frac{-1}{(x-3)^2}$

$\therefore\ f'(x) < 0\ \forall\ x \in R - \{3\}$

$\therefore$ f (x) is monotonocally decreasing function

$\Rightarrow$ f is one-one function.

onto/ into : Let $y \in R - \{1\}$ (co-domain)

Then one element $x \in R - \{3\}$ is domain is such that

$f(x) = y \Rightarrow \frac{x-2}{x-3} = y \Rightarrow x - 2 = xy - 3y$

$\Rightarrow x = \left(\frac{3y-2}{y-1}\right) = x \in R - \{3\}$

$\therefore$ the pre-image of each element of co-domain $R - \{1\}$ exists in domain $R - \{3\}$.

$\Rightarrow$ f is onto. **Ans. [D]**

Ex.14 Function $f : N \to N$, $f(x) = 2x + 3$ is -

(A) one-one onto (B) one-one into

(C) many one onto (D) many one into

Sol. f is one-one because for any $x_1, x_2 \in N$

$x_1 \neq x_2 \Rightarrow 2x_1 + 3 \neq 2x_2 + 3 \Rightarrow f(x_1) \neq f(x_2)$

Further $f^{-1}(x) = \frac{x-3}{2} \notin N$ (domain) when

x = 1, 2, 3 etc.

$\therefore$ f is into which shows that f is one- one into.

Alter

f (x) = 2x + 3

$f'(x) = 2 > 0 \ \forall \ x \in N$

$\therefore$ f(x) is increasing function

$\therefore$ f(x) is one-one function

& $\because$ x = 1 , 2, 3, ...

$\therefore$ min value of f(x) is 2.1 + 3 = 5

$\therefore$ $f(x) \neq \{1, 2, 3, 4\}$

$\therefore$ Co Domain $\neq$ Range

$\therefore$ f(x) is into function **Ans. [B]**

Ex.15 Function $f : R \to R$, $f(x) = x^3 - x$ is -

(A) one-one onto (B) one-one into

(C) many-one onto (D) many-one into

Sol. Since $-1 \neq 1$, but $f(-1) = f(1)$, therefore f is many-one.

Also let, $f(x) = x^3 - x = \alpha \Rightarrow x^3 - x - \alpha = 0$. This is a cubic equation in x which has at least one real root because complex roots always occur in pairs. Therefore each element of co-domain R has pre-image in R. Thus function f in onto .

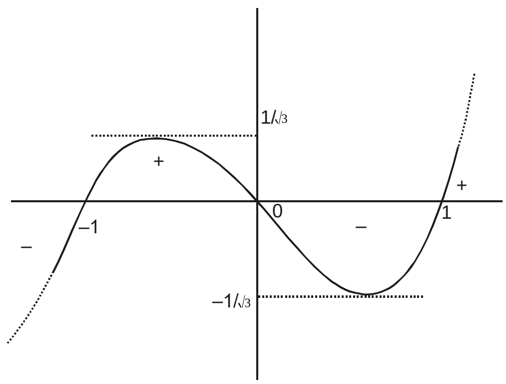
$\therefore$ function f is many-one onto.

Alter

$f(x) = x^3 - x$

= x (x – 1) (x + 1)

graph of f(x) is



from graph function is many one- onto function

Ans. [C]

Ex.16 If $f : R \to R$, $f(x) = 2x - 1$ and $g : R \to R$, $g(x) = x^2 + 2$, then (gof) (x) equals-

(A) $2x^2 - 1$ (B) $(2x - 1)^2$

(C) $2x^2 + 3$ (D) $4x^2 - 4x + 3$

Sol. Here (gof) (x) = g [f(x)] = g (2x – 1)

$= (2x - 1)^2 + 2 = 4x^2 - 4x + 3$. **Ans. [D]**

Ex.17 If $f : R \to R$, $f(x) = 4x^3 + 3$, then $f^{-1}(x)$ equals-

(A) $\left(\frac{x-3}{4}\right)^{1/3}$ (B) $\left(\frac{x^{1/3}-3}{4}\right)$

(C) $\frac{1}{4}(x-3)^{1/3}$ (D) None of these

Sol. Since f is a bijection, therefore f^{-1} exists. Now if f-image of x is y, then $f^{-1}: R \to R$ defined as follows :

$f^{-1}(y) = x \Rightarrow f(x) = y$

But $f(x) = 4x^3 + 3 \Rightarrow y = 4x^3 + 3 \Rightarrow x = \left(\frac{y-3}{4}\right)^{1/3}$

Therefore $f^{-1}(y) = \left(\frac{y-3}{4}\right)^{1/3}$

$\Rightarrow f^{-1}(x) = \left(\frac{x-3}{4}\right)^{1/3}$ **Ans. [A]**

Ex.18 $f(x) = \sqrt{|x-1|}$ and g (x) = sin x then (fog) (x) equals -

(A) $\sin\left\{\sqrt{|x-1|}\right\}$

(B) |sin x/2 – cos x/2|

(C) | sin x – cos x |

(D) None of these

Sol. (fog) (x) = f [g(x)] = f [sin x]

$= \sqrt{|\sin x - 1|}$

$= \sqrt{|1 - \sin x|}$

$= \sqrt{|\sin^2 x/2 + \cos^2 x/2 - 2\sin x/2 \cos x/2|}$

$= \sqrt{|(\sin x/2 - \cos x/2)^2|}$

$= |\sin x/2 - \cos x/2|$ **Ans.[B]**

Ex.19 If f : R → R, f(x) = 2x + 1 and g : R → R, g(x) = x^3,then $(gof)^{-1}(27)$ equals -

(A) –1 (B) 0 (C) 1 (D) 2

Sol. Here f(x) = 2x + 1 $f^{-1}(x) = \frac{x-1}{2}$

and g(x) = $x^3 \Rightarrow g^{-1}(x) = x^{1/3}$

$\because (gof)^{-1}(27) = (f^{-1}og^{-1})(27)$

$= f^{-1}[g^{-1}(27)] = f^{-1}[(27)^{1/3}]$

$= f^{-1}(3) = \frac{3-1}{2} = 1$ **Ans.[C]**

Ex.20 The domain of function f(x) = $\sqrt{2^x - 3^x}$ is -

(A) (– ∞, 0] (B) R

(C) [0, ∞) (D) No value of x

Sol. Domain = {x ; $2^x - 3^x \ge 0$} = {x ; $(2/3)^x \ge 1$}

= x ∈(– ∞,0]

Ans.[A]

Ex.21 The domain of the function

$f(x) = \sin^{-1}\left(\log_2 \frac{x^2}{2}\right)$ is -

(A) [–2, 2] – (–1, 1) (B) [–1,2] – {0}

(C) [1, 2] (D) [–2,2] – {0}

Sol. We know that the domain of $\sin^{-1}x$ is [–1,1]. So for f(x) to be meaningful , we must have

$-1 \le \log_2 \frac{x^2}{2} \le 1$

$\Rightarrow 2^{-1} \le x^2/2 \le 2 \quad x \ne 0$

$\Rightarrow 1 \le x^2 \le 4, x \ne 0$

$\Rightarrow x \in [-2, -1] \cup [1,2]$

$\Rightarrow x \in [-2, 2] - (-1, 1)$

Ans.[A]

Ex.22 The range of function f(x) = $\frac{x^2}{1+x^2}$ is -

(A) R – {1} (B) $R^+ \cup \{0\}$

(C) [0, 1] (D) None of these

Sol. Range is containing those real numbers y for which f(x) = y where x is real number.

Now f(x) = y = ⇒ $\frac{x^2}{1+x^2} = y$

$\Rightarrow x = \sqrt{\frac{y}{(1-y)}}$ ....(1)

by (1) clearly y ≠ 1, and for x to be real

$\frac{y}{1-y} \ge 0 \Rightarrow$ y ≥ 0 and y < 1.

(∵ If y = 2 then $\frac{y}{1-y} = \frac{2}{1-2} = (-2)$ and

$\sqrt{\frac{y}{(1-y)}} = \sqrt{-2} \notin R$)

∴ $0 \le y < 1$

∴ Range of function = (0 ≤ y < 1) = [0,1)

Ans.[D]

Ex.23 If f(x) = cos (log x), then

f(x) f(y) – 1/2 [f (x/y) + f(xy)] is equal to

(A) –1 (B) 1/2

(C) –2 (D) 0

Sol. cos (log x) cos (log y)

$-\frac{1}{2}$[cos (log x/y) + cos (log xy)]

$= \frac{1}{2}$ [cos (log x + log y) + cos (log x – log y)]

$- \frac{1}{2}$ [cos (log x – log y) + cos (log x + log y)]

= 0 **Ans.[D]**

Ex.24 If $f(x) = \frac{2^x + 2^{-x}}{2}$, then f (x + y) . f (x – y) is equal to -

(A) $\frac{1}{2}$ [f (x+ y) + f(x – y)]

(B) $\frac{1}{2}$ [f (2x) + f (2y)]

(C) $\frac{1}{2}$ [f(x+ y). f(x – y)]

(D) None of these

Sol. $f(x + y).\ f(x - y) = \frac{2^{x+y}+2^{-x-y}}{2}.\frac{2^{x+y}+2^{-x-y}}{2}$

$= \frac{2^{2x}+2^{2y}+2^{-2x}+2^{-2y}}{4}$

$= \frac{1}{2}\left[\frac{2^{2x}+2^{-2x}}{2}.\frac{2^{2y}+2^{-2y}}{2}\right]$

$= \frac{1}{2}$ [f (2x) + f(2y)]

Ans.[B]

Ex.25 If f : R → R. f (x) = 2x + |x| , then f (3x) – f (–x) – 4x equals -

(A) f(x) (B) – f(x)

(C) f (–x) (D) 2f(x)

Sol. f (3x) – f (–x) – 4x

= 6x + |3x| – {–2x + | –x|} – 4x

= 6x + 3 |x| + 2x – |x| – 4x

= 4x + 2 |x| = 2 f(x). **Ans.[D]**

Ex.26 If $g(x) = x^2 + x - 2$ and $\frac{1}{2}$ (gof) (x) = $2x^2 - 5x + 2$, then f(x) is equal to -

(A) 2x – 3 (B) 2x + 3

(C) $2x^2 + 3x + 1$ (D) $2x^2 - 3x - 1$

Sol. $g(x) = x^2 + x - 2$

⇒ (gof) (x) = g[f(x)] = $[f(x)]^2 + f(x) - 2$

Given, $\frac{1}{2}$ (gof) (x) = $2x^2 - 5x + 2$

$\therefore \frac{1}{2}[f(x)]^2 + \frac{1}{2} f(x) - 1 = 2x^2 - 5x + 2$

$\Rightarrow [f(x)]^2 + f(x) = 4x^2 - 10x + 6$

⇒ f(x) [f(x) + 1] = (2x – 3) [(2x – 3) + 1]

⇒ f(x) = 2x – 3

Ans.[A]

Ex.27 If f (x) = |x| and g(x) = [x], then value of fog $\left(-\frac{1}{4}\right)$ + gof $\left(-\frac{1}{4}\right)$ is -

(A) 0 (B) 1

(C) –1 (D) 1/4

Sol. fog $= f\left[g\left(-\frac{1}{4}\right)\right]$ f (–1) = 1

and gof $\left(-\frac{1}{4}\right) = g\left[f\left(-\frac{1}{4}\right)\right] = g\left(\frac{1}{4}\right)$ = [1/4] = 0

Required value = 1 + 0 = 1.

Ans.[B]